Physics Done Right, an Attempt

Overview

Table of Contents

0 Introduction

This is a work in progress

This document is very much a work in progress. The current version of it is by no means meant to be published in any form; it is merely the collection of the author's personal drafts and a 'snapshot' of his current ideas. **No proofreading has been performed, propositions might not even hold, and cross-references are not implemented.** Thus, this draft is provided 'as is', with the only intension of illustrating the current 'overall' progress of the work.

0.1 Notation

The following is a quick guide to the notational conventions used in the book, especially in the chapters on classical mechanics.

We will use boldface for vectors, as in $\mathbf{u} \in \mathbb{R}^3$. The length of a vector will be denoted by $u \coloneqq |\mathbf{u}|$. If there already is a boldface letter $\mathbf{u}$ denoting a vector, we might introduce the length u without any explicit remarks. A hat above a vector indicates that the vector is of unit length, as in $\hat{\mathbf{u}}$. By necessity, now, $|\hat{\mathbf{u}}| = 1$. If there already is a vector $\mathbf{u} \neq \mathbf{0}$ we might introduce $\hat{\mathbf{u}} \coloneqq \frac{1}{u}\mathbf{u}$ without any explicit remarks. In a Cartesian coordinate system (x, y, z), the basis vectors are denoted $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. When convenient, and when there is no risk of confusion, we will allow ourselves to identify points with their corresponding radius vectors.

The statement $a \coloneqq b$ means that the value b is assigned to a at the instance of the formula. $a \overset{\text{def}}{=} b$ means that a has previously been assigned the value b, that is, a is equal to b by (an earlier) definition. If f and g are functions of a single variable, $t \in X$, say, then we may write $f(t) \equiv g(t)$ as a shorthand notation for $f(t) = g(t), \forall t \in X$.

There are often two or more frames of reference in the same discussion. In the typical case of two frames $\mathcal{F}1$ and $\mathcal{F}2$, we use a prime to indicate that a variable is with respect to the second frame, $\mathcal{F}2$. For instance, if the coordinates of some object are (x, y, z) relative to $\mathcal{F}1$, we will denote the coordinates of the same object but with respect to $\mathcal{F}2$ by the triplet (x', y', z'). A dot, as in $\dot{x}$, means a derivative with respect to time (or any other *single* parameter), that is, $\dot{x} \coloneqq \frac{d}{dt}x$, and $\dot{x}' \coloneqq \frac{d}{dt'}x'$. If we at some time have a function f of one variable x that does not have the interpretation of 'time', we might *explicitly* introduce the notation f' for the derivative.

We will usually skip the brackets in powers of function values. For instance, $f(x)^n = [f(x)]^n$. We will sometimes make use of the very convenient *Iverson bracket*, which is a map $[\cdot]: \{\text{true, false}\} \to \{0,1\}$ defined by

$$[P] \coloneqq \begin{cases} 1 & \text{if } P \\ 0 & \text{if } \neg P. \end{cases}$$

Finally, we define the *integer interval*

$$[a..b] \coloneqq [a, b] \cap \mathbb{Z}$$

and include 0 in the natural numbers:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\} = \mathbb{Z}^+ \cup \{0\}.$$

1 Classical Mechanics

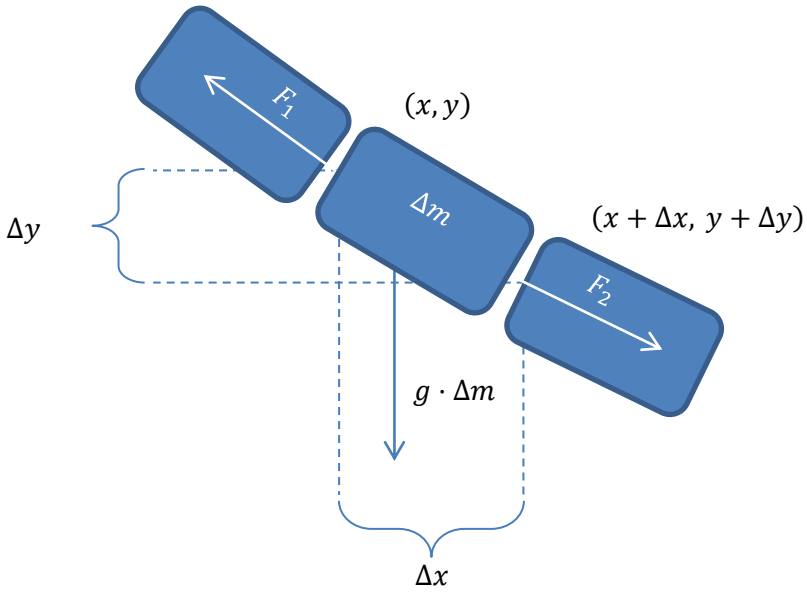


Figure 1. A small part of an ideal chain hanging in a constant force field, such as the field of gravity at the surface of the Earth. In Section 1.4.6 we will prove that such a chain will form the graph of the hyperbolic cosine. A few sections later, we will show that the very same fundamental force will cause planets to orbit stars in elliptical orbits, and comets to fly by stars in hyperbolic trajectories.

1.1 Space and Time

To specify a point in space, we need a *spatial coordinate system*. The general idea is that there is a one-to-one correspondence between physical points in space and elements in $S \coloneqq \mathbb{R}^3$. The procedure is familiar to everyone: One chooses some physical point as the origin. At this point, one imagines three (geometric) basis vectors, which are identified with the standard basis (1,0,0), (0,1,0), (0,0,1) of $\mathbb{R}^3$. Then the general correspondence is obvious: if we get to a particular object from the origin by combining x, y, and $z \in \mathbb{R}$ of these geometric basis vectors using the rules of geometric vector algebra, we say that x, y, and z are the *coordinates* of the object's spatial position, and we associate this spatial position to the element $(x, y, z) \in \mathbb{R}^3$. Almost always one makes sure that the physical basis vectors are of the same length (usually a metre), are perpendicular to each other, and are ordered so they obey the right-hand rule. In this case, we say the system is a *Cartesian* system. For the rest of this section, we will assume all spatial coordinate systems to be Cartesian.

Similarly, to specify the time of some event, we construct a one-to-one correspondence between moments in time and some set T, usually $\mathbb{R}$ or some interval subset of $\mathbb{R}$, called the *timeline* or the *time axis*. The standard procedure is to make a choice about what real-world moment is to correspond to $0 \in T$ (or some other fixed element of T), and then one decides that a change of 1 in time value corresponds to a physical duration of a second (or some other convenient duration).

A *coordinate system* is a spatial coordinate system together with a timeline. Hence, it consists of a choice of spatial origin, three basis vectors, an origin in time, and a time unit. Given a coordinate system, any pointlike event, defined as an event to which we can associate a precise point in space and moment of time, corresponds to a unique element in $T \times S$. This product is sometimes called *spacetime*. [In the literature, one might also see $S \times T$.] Thus, in pre-relativity physics (also called *Newtonian* physics), the concept of spacetime is a very simple and graspable thing.

Let us now turn to the problem of dealing with multiple coordinate systems simultaneously. To begin with, we will assume that all coordinate systems are at rest relative to each other; that is, the origin of each system is stationary (has constant spatial coordinates) relative to every other system. Consider Figure 2.

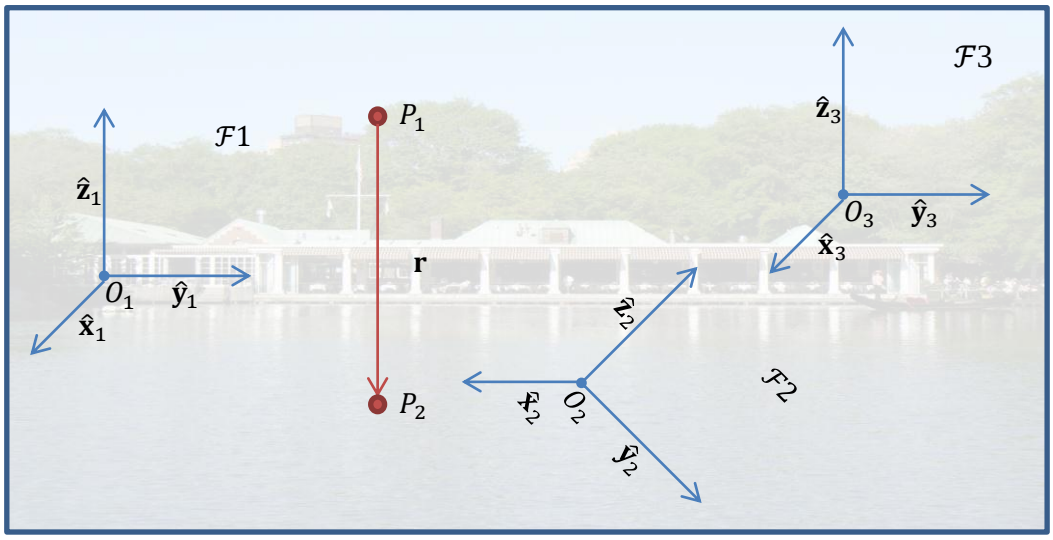


Figure 2. Three spatial coordinate systems in Central Park, New York City.

Here we have indicated three spatial coordinate systems, $\mathcal{F}1$, $\mathcal{F}2$, and $\mathcal{F}3$. Notice that $\mathcal{F}1$ and $\mathcal{F}3$ have the same physical basis vectors. Consider the points P_1 and P_2, at which two insects are hovering at some moment of time; one is at P_1 and the other is at P_2. These points exist independent of any coordinate system. Therefore, we say that they are *geometric points*. The *coordinates* of the points depend on the coordinate system, however. The vector $\mathbf{r}$ is the displacement from P_1 to P_2. This is also an object that exists independent of any coordinate system; hence, it is called a *geometric vector*. The *components* of $\mathbf{r}$, however, depend on the coordinate system. Notice that the components of $\mathbf{r}$ are the same in $\mathcal{F}1$ and $\mathcal{F}3$, because these systems share the same geometric basis vectors: while the coordinates of a point depend on both the choice of origin and the basis vectors, the components of a vector depend only on the basis vectors.

In general, two coordinate systems are in relative motion, that is, the origin of one system is moving with respect to the other system. This is the case, for instance, if one of the systems is stationary on the ground while the other is fixed inside a moving train. When we explicitly include information about the motion of a coordinate system, we call it a (*mathematical) frame of reference*. Restrict for a moment attention to mathematical frames that do not accelerate relative to the ground. If the train is not accelerating, then all frames stationary on the ground and all those fixed inside the train are admissible, as are all other frames moving with constant velocity relative to these. We denote the set of all such reference frames by $\mathcal{F}$. Introduce a relation $\sim$ on $\mathcal{F}$ by declaring that $\mathcal{F}1 \sim \mathcal{F}2$ iff the origin of $\mathcal{F}1$ is at rest relative to $\mathcal{F}2$ at all times, where $\mathcal{F}1, \mathcal{F}2 \in \mathcal{F}$. Clearly, $\sim$ is an equivalence relation on $\mathcal{F}$, and we shall call the equivalence classes *physical frames of reference*.

One might ask if some of the classes $\mathcal{F}/\sim$ contain frames at 'absolute rest'. Naïvely, a child might argue that the ground is in no motion at all, while all the cars, trains, and velocipedes are in motion. However, this is obviously a biased statement, since the Earth is very much in motion around the sun, which in turn, is in motion in its cluster of stars, and eventually the Milky Way galaxy. In addition, the galaxy itself is in motion relative to all other galaxies, and so on. Hence, in out interpretation of Newtonian physics, there is no concept of *absolute* rest, only *relative* rest. In fact, no known experiment can distinguish any preferred frame; all frames in $\mathcal{F}$ are equivalent.

In the discussion in the last paragraphs, we ignored any relative acceleration. Of course, in general, two frames are accelerating relative to each other. Hence, the entire set $\mathcal{F}$ contains only a very special selection of frames, namely, those in uniform motion (i.e., no acceleration) relative to the ground. By the same arguments as in the last paragraph, one might suspect that there is nothing special about $\mathcal{F}$, and that, in fact, *all* mathematical frames of reference are equivalent. This, however, is *not* the case in Newtonian mechanics. Instead, it is postulated that there exists (at least *locally*) a natural collection of mathematical frames that, in some sense, have *vanishing absolute acceleration* (but this shouldn't be taken too literally). Such a frame with 'vanishing absolute acceleration' is called an *inertial* frame, or, less appropriately, a *non-accelerating* frame. It turns out that to a good approximation, a frame of reference attached to the ground of the Earth is inertial. Even more so is a frame in free fall in outer space, which might be considered the 'prototype' of an inertial frame at that location. From now on, we will redefine $\mathcal{F}$ to be the set of all inertial frames with origins in the vicinity of the objects under consideration. The seemingly strange requirement that they need to be locally concentrated in space will be explained later.

But what *is* an inertial frame? How can you tell if a frame is inertial or not? A commonly-stated answer in Newtonian mechanics is that a frame is inertial iff the following implication is valid:

No forces act on a particle ⇒ The particle has zero acceleration.

This is Newton's first law, as we will discuss more thoroughly in the next section. Notice that the frame enters the picture because we measure acceleration with respect to it; only according to some frames does the particle acceleration vanish. It is very important to notice that it is *assumed* that the criterion given above yields the same answer no matter what particle is used to test the frame. (Otherwise the criterion would not make sense.)

Consider a free particle (i.e., a particle on which no forces act) studied using two different mathematical frames of reference, $\mathcal{F}1$ and $\mathcal{F}2$. Let $\mathbf{r}(t)$ be the radius vector of the particle from the origin of $\mathcal{F}1$ at time t, and let $\mathbf{r}'(t)$ be the radius vector from the origin of $\mathcal{F}2$ at the same time. Also, let $\mathbf{R}(t)$ be the instantaneous radius vector of the origin of $\mathcal{F}2$ relative to $\mathcal{F}1$. See Figure 3.

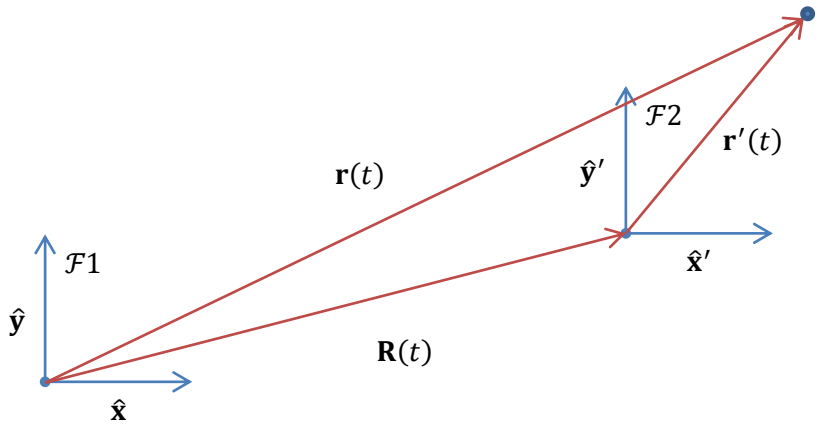


Figure 3. Two frames of reference in relative motion.

Clearly, as geometric vectors,

$$\mathbf{r}(t) = \mathbf{R}(t) + \mathbf{r}'(t), \qquad \forall t \in \mathbb{R}.$$

Differentiating twice with respect to time yields

$$\ddot{\mathbf{r}}(t) = \ddot{\mathbf{R}}(t) + \ddot{\mathbf{r}}'(t), \qquad \forall t \in \mathbb{R}$$

where $\ddot{\mathbf{r}}(t)$, $\ddot{\mathbf{R}}(t)$, and $\ddot{\mathbf{r}}'(t)$ are the acceleration of the particle with respect to $\mathcal{F}1$, the acceleration of the origin of $\mathcal{F}2$ with respect to $\mathcal{F}1$, and the acceleration of the particle with respect to $\mathcal{F}2$, respectively. Assume that $\mathcal{F}1$ is (essentially) inertial, e.g., fixed to the ground of the Earth. Then (↑) holds, and since the particle *is* free, $\ddot{\mathbf{r}}(t) = 0$ by *modus ponens*. Hence,

$$\mathbf{0} = \ddot{\mathbf{R}}(t) + \ddot{\mathbf{r}}'(t), \qquad \forall t \in \mathbb{R}.$$

Therefore,

$$\ddot{\mathbf{R}}(t) = 0 \Leftrightarrow \ddot{\mathbf{r}}'(t) = 0.$$

It is trivial to verify that $\ddot{\mathbf{r}}'(t) = 0 \Leftrightarrow \mathcal{F}2$ is inertial, and therefore (↑) reads, in words,

$$\mathcal{F}1 \text{ and } \mathcal{F}2 \text{ move with constant relative velocity} \Leftrightarrow \mathcal{F}2 \text{ is inertial.}$$

Hence, given *any* inertial frame, all other (nearby) inertial frames, and only those, move with constant velocity relative to the first frame. This motivates our introduction of $\mathcal{F}$ and $\mathcal{F}/\!\sim$: $\mathcal{F}$ is the set of (local) natural frames of reference, and the frames in $\mathcal{F}$ differ only by their relative velocity. There are quite a few subtleties regarding frames and inertial frames, and we will return to these after we have considered some simpler things first.

1.1.1 The Significance of the Physical Frame

The equivalence relation ~ is very useful in Newtonian physics, because many things are similar when analysed with different mathematical frames belonging to the *same* physical frame, but appear different when analysed with different frames belonging to *different* physical frames. The simplest example of this is the velocity vector. Consider a leaf falling from a tree with a constant velocity towards the ground. When analysed in a frame fixed on the ground, the leaf appears to be falling with the velocity vector **v** shown in Figure 4.

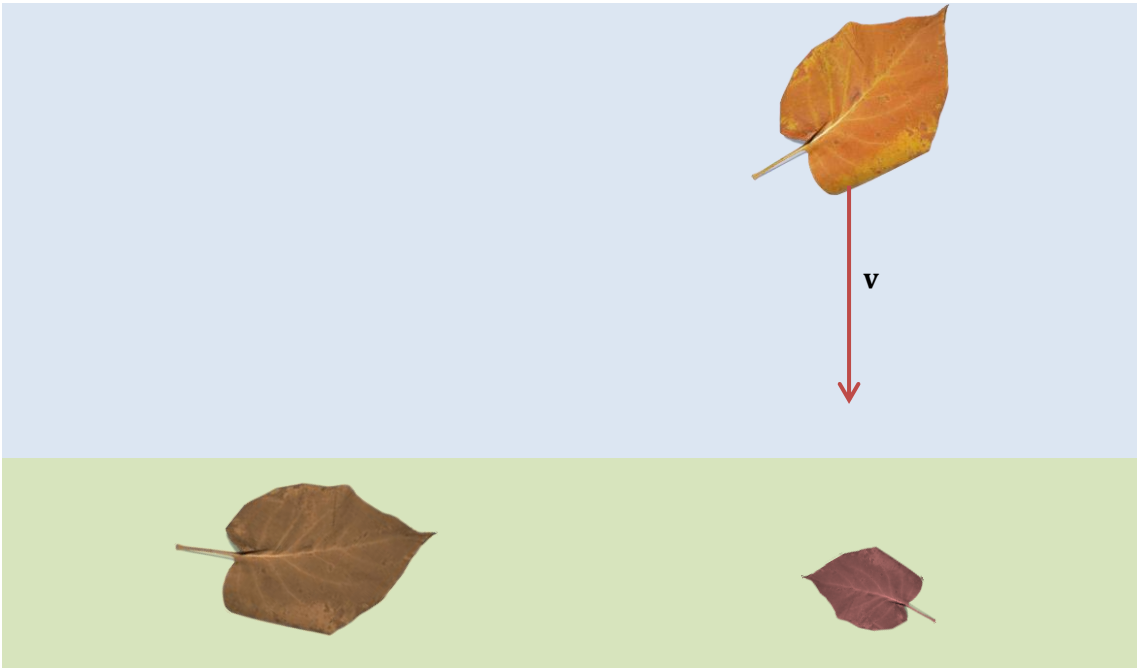


Figure 4. A falling leaf as seen from the ground.

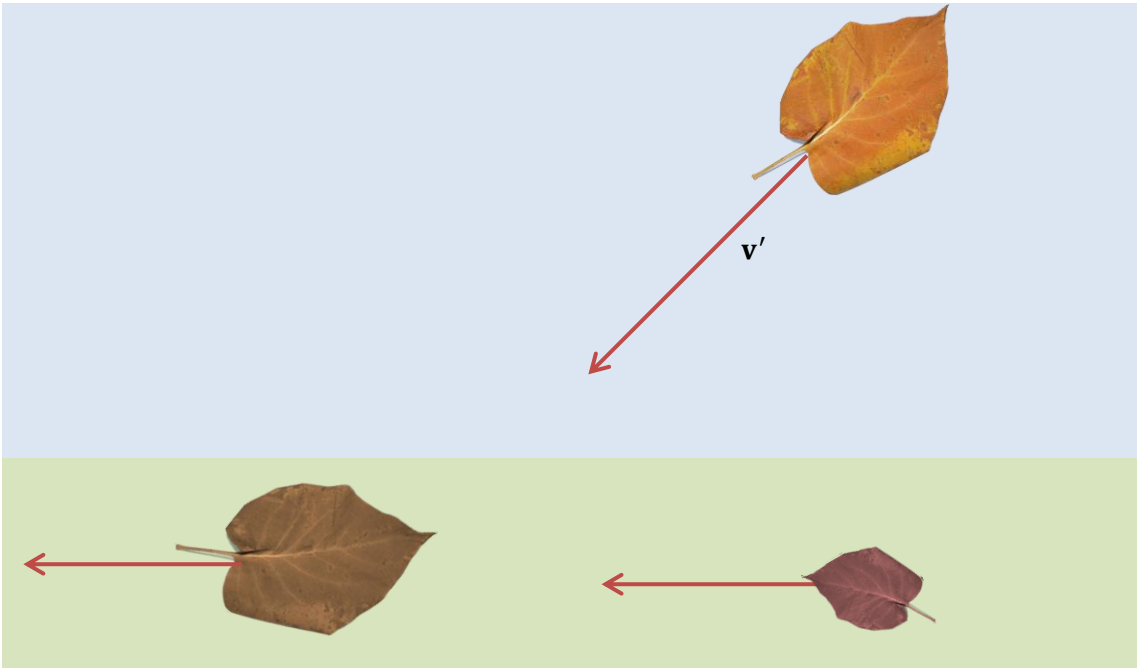


Figure 5. The same falling leaf as seen from a train.

Now, consider the same situation, but observed from a train moving to the right with the same speed as the leaf is falling (relative to the ground). From the point of view of a passenger on the train, the leaf's velocity vector is now $\mathbf{v}'$, as shown in Figure 5. Hence, as seen from the ground, the leaf's velocity vector points towards the small, red, leaf on the ground, while, as seen from the train, the 'same' velocity vector points towards the big, brown, leaf on the ground. The conclusion is that the velocity vector of an object *is not a geometric vector*, but depends on the physical frame of reference. Inside a *given* physical frame, however, the vector looks the same no matter what mathematical frame is used – only the components of the vector differ between mathematical frames. Thus, only if we restrict our attention to a specific physical frame of reference, we may allow ourselves to think of a velocity vector as a geometric entity.

The reader may recall Figure 2. The displacement vector $\mathbf{r}$ shown there clearly *is* a geometric vector. Now, ignore the insect located at P_2. Instead, imagine that $\mathbf{r} = \mathbf{v}$ is the *velocity* of the insect initially at P_1. Then, as shown above, $\mathbf{v}$ is not a geometric vector. Only according to some observers is it perpendicular to the surface of the lake! Now, let $\mathbf{r} = \mathbf{a}$ be the *acceleration* of the insect, instead. Then, to some extent, 'normality' is restored, because acceleration vectors *are* independent of frames of reference as long as we restrict attention to inertial frames, that is, to frames in $\mathcal{F}$. Thus, accelerations vectors – and therefore force vectors – may be thought of as geometric entities if we only consider inertial frames. You already knew this: surely the force of gravity, and the ensuing acceleration of a cup of coffee, points straight to the ground no matter if you stand on the ground or in a train moving with constant velocity.

The observations made above are formalised in the *Galilean transformation*, which relates coordinates between two inertial frames. Let $\mathcal{F}1, \mathcal{F}2 \in \mathcal{F}$. Let $\mathbf{r}(t)$ be the radius vector of an insect from the origin of $\mathcal{F}1$ at time t and let $\mathbf{r}'(t)$ be the radius vector from the origin of $\mathcal{F}2$ at the same time. Also, let $\mathbf{R}(t)$ be the radius vector from the origin of $\mathcal{F}1$ to the origin of $\mathcal{F}2$ at this time. See Figure 6.

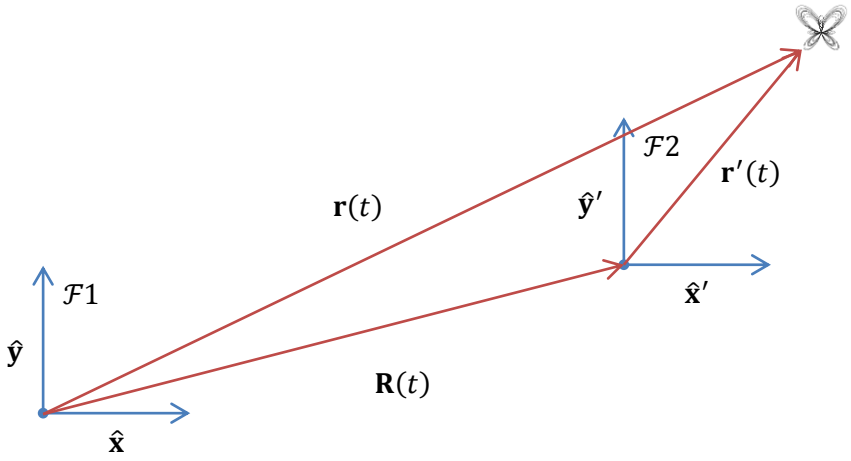


Figure 6. Two coordinate systems in relative motion.

The Galilean transformation is simply

$$\mathbf{r}(t) = \mathbf{R}(t) + \mathbf{r}'(t), \qquad \forall t \in \mathbb{R}.$$

Differentiating with respect to time,

$$\dot{\mathbf{r}}(t) = \dot{\mathbf{R}} + \dot{\mathbf{r}}'(t), \qquad \forall t \in \mathbb{R}$$

where $\dot{\mathbf{r}}(t)$ is the velocity vector of the insect relative to $\mathcal{F}1$, $\dot{\mathbf{r}}'(t)$ is the velocity relative to $\mathcal{F}2$, and $\dot{\mathbf{R}}$, which is independent of time, is the velocity of the origin of $\mathcal{F}2$ relative to $\mathcal{F}1$. Hence, if $\mathcal{F}1 \nsim \mathcal{F}2$, $\mathbf{R} \neq \mathbf{0}$ and so $\dot{\mathbf{r}}(t) \neq \dot{\mathbf{r}}'(t)$. Differentiating once more,

$$\ddot{\mathbf{r}}(t) = \ddot{\mathbf{r}}'(t), \qquad \forall t \in \mathbb{R}$$

because $\ddot{\mathbf{R}} = \mathbf{0}$. Thus, the acceleration vector is independent of the choice of inertial frame; only the components can differ.

Finally, let us introduce some standard nomenclature: two frames $\mathcal{F}1$ and $\mathcal{F}2 \in \mathcal{F}$ are in *standard configuration* iff

(1) they have the same geometric basis vectors,

(2) the velocity of the origin of $\mathcal{F}2$ relative to $\mathcal{F}1$ is a non-negative multiple of the first basis vector of $\mathcal{F}1$, and

(3) at $t = 0$ also $t' = 0$ and at this time, their spatial origins coincide.

In this case the Galilean transformation reads

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

where $v = |\mathbf{v}| = \left|\dot{\mathbf{R}}\right|$ is the relative speed between the frames.

1.1.2 The Concept of Inertial Frames

You might feel a bit uneasy about the introduction of a preferred set of frames, such as $\mathcal{F}$. If so, you can rest assured because this concept is abandoned in the theory of General Relativity. For now, let us investigate the concept of inertial frames from a sceptical point of view without brining (too much) relativity into the discussion.

In every-day situations, it is rather easy to distinguish an inertial frame from a non-inertial frame. The traditional example is a person standing on the ground and in a bus (well, not simultaneously, obviously!). On the ground, we know that the force of gravity acts in the direction towards the centre of the Earth, and an equally large upward-pointing normal force is exerted on the person from the ground. Hence, the net force is zero, and the person is not accelerating relative to the ground. Hence, the ground is inertial. The very same experiment can be performed in a bus moving with constant velocity relative to the ground. Still no net force, and still no acceleration of the person. Hence, the bus is also an inertial frame. However, if the bus begins to accelerate, suddenly the person seems pushed towards the rear end of the vehicle, although no additional force is acting on him. Now the bus is *not* an inertial frame.

What is the issue, then? Well, one issue is the following: How do we *really* know that no additional force (pointing towards the rear end of the bus) acting on the body set in at the time when the bus began to accelerate? The classical answer, of course, is that we have a very complete theory of classical mechanics, and we have classified all the forces acting in it. We have managed to create a theory that works exceedingly well, and is self-consistent. Hence, we can say for sure that, within the theory of Newtonian physics, no additional force kicks in.

A second issue is this: to test whether a frame is inertial or not, we need a particle on which the net force is zero. In practice, we cannot find any such particle, because all massive particles are

affected by the long-range force of gravity from every other massive particle in the universe. Why don't we notice this effect, for instance, when we are standing in a bus? The answer is that the force of gravity affects *all* massive particles, including the bus, the persons in the bus, and every atom of the Earth itself, in exactly the same way.

Let us consider a third issue. Consider a container in free fall in the vicinity of the Sun. An astronaut inside the container, but not aware of its location in the universe, will observe that the container behaves exactly like an inertial frame of reference. For example, a ball carefully positioned at some place inside the container will remain there until provoked by some force known to the astronaut. Of course, you can perform the same experiment in a container in free fall close to the Earth. Both these frames will be found to be inertial. But they are clearly accelerating relative to each other (one is accelerating towards the Sun, the other towards the Earth), thus contradicting the result that all inertial frames are in uniform motion relative to each other. What is the cause of this contradiction?

The answer may seem obvious. *Neither* of the frames are 'really' inertial frames, because, clearly, they are being accelerated towards the sun and the Earth, respectively. Hence, the ball inside the first container is affected by a strong gravitational force, and, indeed, is accelerating towards the sun. But since this applies equally to every atom of the container, including the matter that makes up the astronaut himself, he cannot know this. This seems to resolve the issue, because, since neither frame is inertial, nothing says that they should be in uniform motion relative to each other.

But this raises another question. By the argument given in the last paragraph, a reference frame attached to some material body (a planet, a comet, a spacecraft, …) can *never* be inertial, because every massive particle in the entire universe is affected by the force of gravity; hence, it is accelerating. In fact, the 'explanation' given in the last paragraph is almost as naïve as the explanation given by the child stating that the ground of the Earth is an 'absolute rest' frame. Indeed, one might argue, if we cannot use experiments to determine whether a frame is inertial or not, then the concept of 'inertial frame' doesn't even belong to physics!

A well-known (but outdated) 'resolution' of the issue consists of actually postulating that there exists a frame of 'absolute rest', perhaps by defining this frame to be stationary relative to the (centre of mass of the?) fixed stars. Then every frame moving with constant velocity relative to this frame is defined to be inertial. This 'resolution' is far from satisfactory. For instance, what about other galaxies? These are very much accelerating relative to our galaxy.

The modern resolution of the issue can be approached by studying the statement and proof of the result given earlier (c.f. page 13), that two inertial frames must be in uniform motion with respect to each other. [Given that we have found a 'counterexample' for this 'theorem', it is indeed very natural to reinvestigate it!] The problem is the assumption that *any particle* can be used to verify the criterion (↑) that the frame is inertial. In reality, two different particles *can* yield different outcomes. If they are close to each other, the difference in their acceleration due to the field of gravity is small, and the assumption is valid (to an extremely good degree), but if the distance becomes 'astronomical', differences are to be expected, as in this case. In the derivation of the result on page 13 we used the *same* particle to test 'inertia' in both frames, but if the frames are very far away, then at least one of the frames will be very far from the particle. Even if the frame at the Earth seems inertial when tested by the ball in its container, it might not (in fact, does not) seem inertial when tested using the ball in the other container, near the sun.

What is the cause of the difference? The answer is the big difference in gravitational field between the two locations. To see this, consider again the container near the sun. Indeed, this seems inertial when tested by the ball inside the container. If we add another ball to the container, then both will yield the same answer (not exactly, but a human eye wouldn't detect any difference). Now imagine that the second ball is moved a few kilometres away from the sun, towards the Earth. Still no major difference; the force acting on the ball is essentially the same as before. Move it even further away. Eventually, it will come close to the Earth, and now the acceleration will be completely different. It will point towards to the Earth, and, obviously, it will accelerate relative to the container at the sun.

The conclusion is the following: the test (↑) *is* essentially independent on the location of the test particle, as long as we restrict the attention to a region in space in which the gravitational field is roughly constant. Of course, this includes all every-day scenarios in homes and laboratories on Earth. If one needs to investigate regions in space with significant changes of gravitational field, one needs to take the sources of these changes into account, like the sun and the Earth.

The 'modern' definition of an inertial frame is a frame with its origin in free fall, that is, the origin moves exactly like a small test particle would if it were only affected by the field of gravity, and no other forces. Even though this frame is accelerating, an astronaut in a container fixed at the origin would perceive the frame as inertial. Again, this is because the container, the astronaut, and his ball – indeed, every massive particle there is – is affected by the very same field of gravity: if the container is so small that we can neglect changes in the field of gravity inside it, then all these objects will have the exact same acceleration because of gravity.

You might get the feeling that the force of gravity is very different from the other forces of nature. We will get back to this point, which, in fact, is the starting point of General Relativity.

1.2 Newton's Laws

Classical mechanics is the foundation of physics, and Newton's laws are the postulates of classical mechanics. Hence, their significance is paramount. The laws are due to Sir Isaac Newton in his *Philosophiæ Naturalis Principia Mathematica*, first published in 1687, and applies to matter particles in space. By 'matter particle', we mean a body of matter with such a small extent that we can neglect deformation and rotation of the body. In order to formulate the three laws, we need first to define the quantities involved. It is understood that we work in some mathematical frame $\mathcal{F}1 \in \mathcal{F}$. In this frame, we associate with a particle a position vector $\mathbf{x} = (x, y, z)$ which is allowed to change with time, if the particle is moving relative to $\mathcal{F}1$. The time derivative $\mathbf{u} \coloneqq \dot{\mathbf{x}}$ is called the *velocity* of the particle, and the second derivative $\mathbf{a} \coloneqq \ddot{\mathbf{x}}$ is called the *acceleration*. A particle moving with zero acceleration is moving with constant velocity, and, consequently, along a straight line with direction $\hat{\mathbf{u}}$ and constant speed u. Each matter particle has a well-defined property called its *mass* that is constant in time. In classical mechanics, the mass is a measure of matter content; indeed, *any* material body can be thought of as being composed of N 'standard matter particles' of equal mass m, and thus has mass Nm. More quantitatively, we will see that Newton's second law of motion postulates that the mass of an object is a measure of the object's resistance to change its velocity. Furthermore, Newton's law of universal gravitation postulates that the masses of two objects determine the magnitude of the force of gravity between them; we will get back to this law.

After mass, the most important concept in Newtonian physics is *force*. A *force* is a vector quantity that, as postulated by Newton's second law of motion, tries to alter the velocity of a particle. Such a force is said to 'act' on the particle. If N forces $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_N$ act on a particle, the net effect will be indistinguishable from that caused by a single force $\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i$ acting on the particle.

Now, let us formulate Newton's laws. In any inertial frame,

1. $\mathbf{F} = \mathbf{0} \Rightarrow \mathbf{a} = \mathbf{0}$. If the net force of a particle is zero, then its velocity is constant. That is, if 'nothing happens' to a particle, it is not its position that is constant, but its derivative, the velocity.

2. $\mathbf{F} = m\mathbf{a}$. The net force on a particle is equal to the product of its mass and its acceleration. Notice that the first law is a consequence of the second law.

3. If a particle A affects B with a force $\mathbf{F}_{AB}$, then B affects A with a force $\mathbf{F}_{BA} = -\mathbf{F}_{AB}$ of equal magnitude but opposite direction.

Figure 7. Illustration of Newton's third law.

There are two ways of thinking about force. *Either*, one might consider force as a geometric entity in space, that is, as a geometric vector, *or* one might consider the force to be defined in terms of the measurable acceleration of a particle, using Newton's second law, as seen from some frame.

APPROACH 1: The most naïve approach is to consider force as a geometric vector. In this approach, the force is a geometric entity that exists independent of frame of reference. Consider an apple hovering at rest inside the ISS. If someone begins to push the apple, then the force on the apple is a well-defined geometric vector **F**, that – at least in principle – *every* observer (inertial or not) can determine. This approach also goes hand-in-hand with the concepts of the gravitational and electrostatic force fields, which are (literally) defined as geometric vector fields. The main issue with this approach is that there is no direct way of measuring the force. Still, this is the most common approach.

APPROACH 2: The other approach is to define the force as the product of mass and acceleration of an object. Defined this way, the force is (in principle) very easy to measure in any frame of reference, for the acceleration is easily measurable (in principle).

The situation is further complicated by the fact that the concept of 'mass' itself might be hard to define in a satisfying manner. If one thinks that the concept of force is more natural than the concept of mass, then one can postulate the existence of forces, and then use Newton's second law to *define* the mass of a particle.

1.2.1 The Invariance of Newton's Laws

We will 'show' the invariance of Newton's laws under Galilean transformations. This is not as straightforward as one might suspect, mainly because the concept of 'force' lacks a clear, intrinsic, definition, as discussed above.

First, assume that $\mathcal{F}1 \in \wp\mathcal{F}$ and $\mathcal{F}2 \in \wp\mathcal{F}$ are two mathematical frames in the *same* physical frame $\wp\mathcal{F}$. (1) Clearly, a *translation* of the time variable will not change any of the laws of physics, since the origin of time is arbitrary. (2) A pure *rotation* corresponds to a mere change of vector basis. That is, observers will find the same geometric vectors, but their components will vary. The laws of physics will not change, because space itself is assumed isotropic in Newtonian mechanics. (3) A *translation* of the origin will not affect the physics, since the origin is arbitrary; in other words, space itself is assumed homogeneous.

Now let $\mathcal{F}1 \in \wp\mathcal{F}1$ and $\mathcal{F}2 \in \wp\mathcal{F}2$ be two reference frames in two *distinct* physical frames $\wp\mathcal{F}1$ and $\wp\mathcal{F}2$. Assume that Newton's laws are valid in $\mathcal{F}1$. We want to show that they are valid in $\mathcal{F}2$ as well. Since the first law is a consequence of the second law, we will not treat this separately. Therefore, let A be an object that is found to obey the second law

$$\mathbf{F} = m\mathbf{a}$$

in $\mathcal{F}1$. In Newtonian physics, the mass m of an object is an intrinsic property of the object, and therefore, observer-independent. As shown above, the acceleration vector $\mathbf{a}'$ of A as perceived by an observer in $\mathcal{F}2$ is equal to the acceleration vector as perceived from $\mathcal{F}1$, that is, $\mathbf{a}' = \mathbf{a}$ [but the *components* of this vector depend on the vector basis of the frame]. As a geometric vector, the right hand side of (↑) is thus the same in both frames. But what do our observers in the two frames have to say about the forces on A?

According to the *first* approach given above, we consider the force as given by a force field, e.g. the gravitational field from a massive particle, and argue that this is a geometric object and independent of physical frame. Then it follows that the LHS of (↑) is invariant too, and we have shown that Newton's second law is invariant under a Galilean transformation. According to the *second* approach given above, 'force' is *by definition* precisely the RHS of (↑), and so Newton's second law is valid in $\mathcal{F}2$, and, in addition, we have now *shown* that the force is found to be the

same in any physical frame. That is, since the acceleration, as a geometric vector, is the same in any inertial frame, it *follows* that the force, as a geometric vector, is also the same in any inertial frame. Hence, in practice, the difference between the two approaches is not very important.

1.2.2 Non-Inertial Frames

Newton's laws are only valid in inertial frames. To aid the discussion about non-inertial frames, we will introduce a new concept, to be used only in this subsection. Recall that we defined a 'physical frame' as an equivalence class of (mathematical) frames of reference, the origins of which are at rest (that is, have zero geometric *velocity*) relative to each other. We now define a 'super-physical frame' as an equivalence class of (mathematical) frames of reference, the origins of which have zero geometric *acceleration* relative to each other at every time. Then, intuitively, all inertial frames constitute one super-physical frame. Indeed, two inertial frames are moving relative each other with constant velocity. If another frame $\mathcal{F}2$ is accelerating with constant acceleration $\mathbf{a} = 5\hat{\mathbf{x}}$ relative to an inertial frame $\mathcal{F}1$, then this frame belongs to a different super-physical frame, together with frames that are rotated or displaced in space or velocity relative to $\mathcal{F}2$.

It is easy to show that, if Newton's laws are valid in one super-physical frame (such as the super-physical frame of inertial frames), then they cannot be valid in any other such frame. The proof depends on which approach we choose to employ when talking about forces.

In APPROACH 1, the force on an object is an intrinsic, geometric, property, and so it is the same in every frame. Let $\mathbf{F}$ be the force on an object A. Let $\mathcal{F}1 \in \mathcal{S}\wp\mathcal{F}1$ and $\mathcal{F}2 \in \mathcal{S}\wp\mathcal{F}2$ be two frames in *different* super-physical frames. Assume that Newton's laws are valid in $\mathcal{F}1$. Then the acceleration is

$$\mathbf{a} = \frac{1}{m}\mathbf{F}$$

where $\mathbf{F}$ is the force on A. In $\mathcal{S}\wp\mathcal{F}2$, the acceleration is $\mathbf{a}'$, and since this is a *different* super-physical frame, $\mathbf{a} \neq \mathbf{a}'$. But since the force is an intrinsic, geometric, property,

$$\mathbf{a}' = \frac{1}{m}\mathbf{F}$$

which is a contradiction.

Let us consider the same set-up but using APPROACH 2. Assume that the universe is empty except for one, single, particle, which we call A, and which is at rest relative to $\mathcal{F}1$. In this frame, $\mathbf{a} = \mathbf{0}$ and therefore, *by definition*, $\mathbf{F} = \mathbf{0}$. Relative to $\mathcal{F}2$, the acceleration of A is $\mathbf{a}' \neq \mathbf{0}$, and so, by definition, $\mathbf{F}' \neq \mathbf{0}$. Hence, even though the particle is alone in the universe, 'something' is producing a force on it. According to Newton's third law, then A must affect this 'something' by the force $-\mathbf{F}' \neq \mathbf{0}$. But this 'something' doesn't exist! Maybe you could rescue the theory by making some more or less reasonable explanations, but this is a bit too strange to seem natural, considering this chapter is devoted to *classical* mechanics.

1.3 The Fundamental Forces

According to the standard models of modern physics, there are four fundamental forces in nature, namely, gravitation, electromagnetism, the strong nuclear force, and the weak nuclear force. At the end of the seventeenth century, Newton formulated the law of universal gravitation that is one of the cornerstones of classical mechanics; it gives the gravitational force $\mathbf{F}$ experienced *by* a body B_2 of mass $m_2 > 0$ *due to* another body B_1 of mass $m_1 > 0$ a distance r away. If $\hat{\mathbf{r}}$ is a unit vector pointing from B_1 to B_2, then[1]

$$\mathbf{F} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

where G is a constant[2]. (Notice that gravity is an unquestionably *attractive* force.) Therefore, by Newton's second law of motion, the acceleration of B_2 is[3]

$$\mathbf{a} = \frac{1}{m_2}\mathbf{F} = -G\frac{m_1}{r^2}\hat{\mathbf{r}}$$

and is independent of the mass m_2. This is known as Galileo's law: if you drop a feather and an iron weight from the same height above the surface of the Earth, then – neglecting all forces besides gravity (predominantly air resistance) – they will stay next to each other during the fall, and they will hit the ground simultaneously.[4]

Now, consider two electrically charged bodies B_1 [the source of the field] and B_2 [the test particle] with charges $q_1 \in \mathbb{R}$ and $q_2 \in \mathbb{R}$, respectively. Then Coulomb's law gives the electrostatic force[5]

$$\mathbf{F} = k\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$

on the test charge B_2 due to the source charge B_1. This force clearly has the exact same form as in the case of gravity; instead of the masses of the particles, the force is now proportional to their charges. However, notice now that the force is *repulsive* if $\operatorname{sgn} q_1 = \operatorname{sgn} q_2$ [and, of course, neither charge is zero]. More importantly, the acceleration of B_2 is now

$$\mathbf{a} = \frac{1}{m_2}\mathbf{F} = k\frac{q_1 q_2}{m_2 r^2}\hat{\mathbf{r}}$$

and thus it depends crucially on both the mass m_2 and the charge q_2 of the test body. The strong and weak nuclear forces behave more like the electrostatic force than the force of gravity in this respect {{kb}}, and we therefore conclude that gravity is a rather special force, since it is propor-

[1] We often consider B_1 to be a fixed *source* of the gravitational field, such as a star or a planet, and B_2 to be a small 'test particle' experiencing this field. In this case, m_1 is called the 'active gravitational mass', and m_2 the 'passive gravitational mass'. Of course, we know that to experimental accuracy, these two quantities are the same for any object, which is also implied in the law of universal gravitation, since we only speak of the 'mass'.

[2] In SI units, $G = 6.674 \cdot 10^{-11}\ \mathrm{Nm^2kg^{-2}}$.

[3] The mass entering in Newton's second law is called the 'inertial mass'. Again, to experimental accuracy, the 'inertial mass' of body is equal to both its 'gravitational masses'. Hence, we may simply speak of 'the mass' of an object.

[4] Even today, you might hear that this is counterintuitive. Personally, I have never quite understood why. Indeed, if you take a thousand feathers, arrange them in a simple cubic array of spacing $\epsilon > 0$, and then release them at the same time, they will all fall with the feather's acceleration, irrespective of the precise value of ϵ. But as ϵ is decreased, you will eventually obtain the density of an iron weight. Therefore, an iron weight has to fall with the same acceleration as a feather. *Quod erat demonstrandum.*

[5] In SI units, $k = 1/4\pi\epsilon_0 = 8.988 \cdot 10^9\ \mathrm{Nm^2/C^2}$ where 1 C is the Coulomb unit of electric charge.

tional in strength to the inertial mass of Newton's second law. This suggests that there is something very special about gravity.

1.4 Simple Examples of Kinematics

We will use Newton's laws to investigate the kinematics of a number of simple examples of physical systems.

1.4.1 Two Massive Bodies

Let A and B be two bodies with masses m_A and m_B, initially (at time $t = 0$) at rest at $(0, 0, 0)$ and $(r_0, 0, 0)$ relative to an inertial frame $\mathcal{F}1 \in \mathcal{F}$, where the initial distance $r_0 > 0$. Due to the force of gravity, these will accelerate towards each other, with an ever-increasing acceleration.

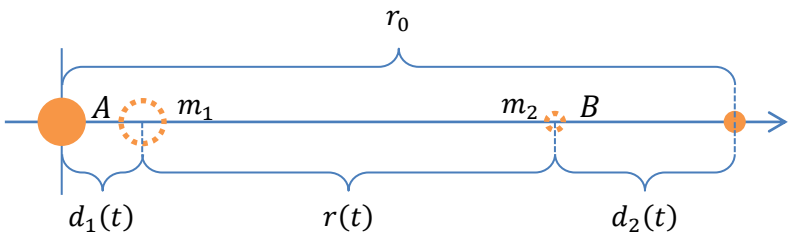


Figure 8. Two bodies approaching each other due to the force of gravity.

Let $d_1(t)$ and $d_2(t)$ be the distances travelled by A and B at time t, respectively, and let $r(t)$ be the distance between A and B at this time. It follows that, for all times prior to collision,

$$d_1(t) + r(t) + d_2(t) = r_0.$$

Differentiation with respect to time (twice) yields

$$\ddot{d}_1(t) + \ddot{r}(t) + \ddot{d}_2(t) = 0.$$

But Newton's second law and the law of gravitation combines to yield

$$m_1\ddot{d}_1(t) = \frac{Gm_1m_2}{r(t)^2}, \qquad m_2\ddot{d}_2(t) = \frac{Gm_1m_2}{r(t)^2}.$$

Therefore,

$$\frac{Gm_2}{r(t)^2} + \ddot{r}(t) + \frac{Gm_1}{r(t)^2} = 0.$$

Rearrange the terms to obtain

$$\ddot{r}(t) + \frac{k}{r(t)^2} = 0$$

where

$$k \coloneqq G(m_1 + m_2).$$

Thus, the problem has the precise mathematical formulation

$$r'' + kr^{-2} = 0, \qquad \begin{cases} r(0) = r_0 \\ r'(0) = 0 \end{cases} \quad (k, t > 0)$$

and it is perfectly sensible to solve it numerically, especially since the solution $t \mapsto r(t)$ probably cannot be expressed in terms of elementary functions. However, one can show that the bodies will collide at time $t = t_{\text{collision}}$ where

$$t_{\text{collision}} := \frac{\pi}{2\sqrt{2k}} r_0^{3/2}.$$

As a concrete example, let $m_1 = m_2 = 2$ kg and $r_0 = 1$ m. Then $t_{\text{collision}} = 18.9$ hours. A graph of $t \mapsto r(t)$ for $t \in [0, t_{\text{collision}}[$ is shown below.

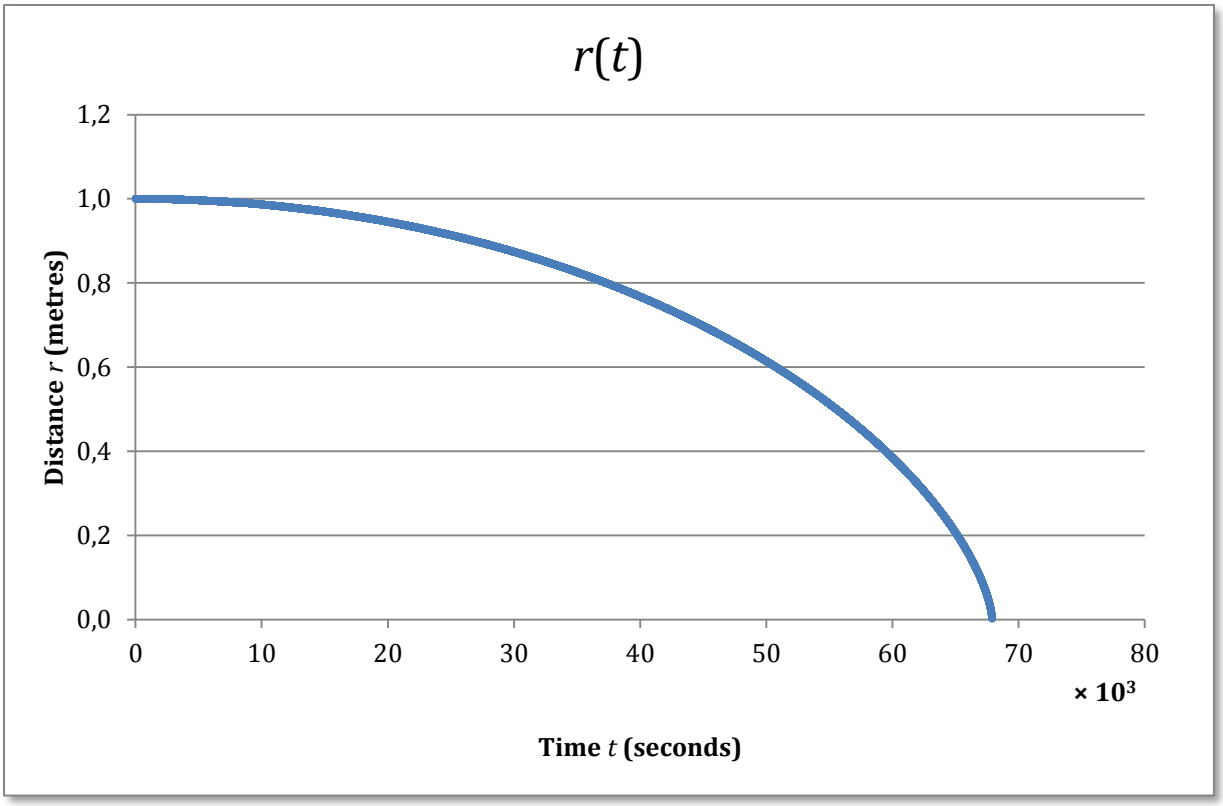


Figure 9. Distance between two gravitationally interacting bodies versus time.

A full exact treatment of the ODE (↑) is given in Appendix A.2.

1.4.2 Projectile Motion

Assume that a ball of mass m is thrown with an initial velocity $\mathbf{v}_0 = (v_{0x}, v_{0y})$ by an experimental physicist or a British actor. We neglect air resistance, so the only force acting on the ball is the force $\mathbf{F}_g = -mg\hat{\mathbf{y}}$ of gravity, where g is the constant *acceleration due to gravity*, as we will discuss in a later section.

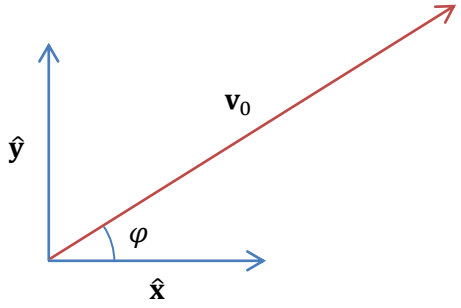


Figure 10. The initial velocity of a thrown ball.

Put the origin at the physicist/actor, more precisely at the point of the ball as it leaves the hand, and denote by $\mathbf{r}(t) = \big(x(t), y(t)\big)$ the position of the ball at time t. Then Newton's second law reads

$$m\ddot{\mathbf{r}} = \mathbf{F}_g,$$

or – explicitly –,

$$\begin{aligned} m\ddot{x} &= 0 \\ m\ddot{y} &= -mg. \end{aligned}$$

Thus, the horizontal component $\dot{x}$ of the velocity is constant, namely, $\dot{x}(t) = v_{0x}$. Therefore

$$x(t) = v_{0x}t$$

since $x(0) = 0$. Now

$$m\ddot{y} = -mg \Rightarrow \ddot{y} = -g \Rightarrow \dot{y} = v_{0y} - gt \Rightarrow y(t) = v_{0y}t - \frac{1}{2}gt^2 = t\left(v_{0y} - \frac{1}{2}gt\right)$$

since $\dot{y}(0) = v_{0y}$ and $y(0) = 0$. The ball hits the ground[6] when $t > 0$ and $y(t) = 0$, that is, when

$$t = t_{\text{impact}} \coloneqq \frac{2v_{0y}}{g}.$$

At this time, the ball has travelled a horizontal distance

$$x_{\text{impact}} \coloneqq x\big(t_{\text{impact}}\big) = \frac{2v_{0x}v_{0y}}{g}.$$

Assume that the angle between the ground ($\hat{\mathbf{x}}$) and $\mathbf{v}_0$ is $\varphi \in [0, \pi/2]$ so that

$$\begin{aligned} v_{0x} &= v_0 \cos\varphi \\ v_{0y} &= v_0 \sin\varphi. \end{aligned}$$

Then

$$x_{\text{impact}}(\varphi) = \frac{2v_0^2}{g}\cos\varphi \sin\varphi = \frac{v_0^2}{g}\sin 2\varphi.$$

Clearly, for any given initial speed v_0, the longest throw (measured horizontally) is obtained by directing it 45° above the ground. What is the maximum height of the ball? The maximum of $y(t)$ is clearly obtained when $\dot{y}(t) = 0$. From (↑) and (↑), this occurs precisely when

$$t = t_{\text{top}} \coloneqq \frac{v_{0y}}{g} = \frac{1}{2}t_{\text{impact}},$$

and the maximum height is

$$y\big(t_{\text{top}}\big) = \frac{v_{0y}^2}{2g} = \frac{v_0^2 \sin^2\varphi}{2g}.$$

Not surprisingly, the maximum height is achieved when the throw (initial velocity) is purely vertical, i.e., when $\varphi = \pi/2$. Finally, notice that (↑) combined with (↑) yields the path

$$y(x) = \tan\varphi \cdot x - \frac{g}{2v_{0x}^2}x^2.$$

[6] Strictly speaking, it hits the surface $y = 0$ which is slightly above the ground unless the ball is thrown exactly from the ground by the physicist/actor.

That is, the path is a *parabola*; this motivates the Swedish word 'kastparabel'.

1.4.3 Circular Motion

Assume that a particle of mass m is circling the origin of some xy plane. What does the force $\mathbf{F}$ on the particle have to look like? Well, if the radius of the orbit is $r > 0$, then

$$\mathbf{r}(t) = (r\cos\omega t\,, r\sin\omega t)$$

for some constant ω, the angular frequency of the particle. This implies

$$\ddot{\mathbf{r}}(t) = (-r\omega^2\cos\omega t\,, -r\omega^2\sin\omega t) = -\omega^2\mathbf{r}(t).$$

That is, the acceleration is always directed towards the origin (that is, the centre of the circle) and has constant magnitude

$$\ddot{r} = \omega^2 r.$$

Since the angular speed (=angular frequency) of the particle is ω, the speed of the particle is $v \coloneqq r\omega$. Thus

$$\ddot{r} = \frac{v^2}{r}.$$

The acceleration of a particle travelling in a circular orbit is called the *centripetal acceleration*. The (net) *force* on such an object [with mass m], the *centripetal force*, therefore has magnitude

$$F = \frac{mv^2}{r}$$

and is always directed towards the centre of the circle. Of course, the physical nature of this force can be of any kind, such as gravitational and electromagnetic (e.g., a contact force).

1.4.4 The Ideal Spring – Simple Harmonic Motion

Assume that you put an ideal spring of length L (when not stretched) along the x axis with one end fixed to the origin $x = 0$ while the other (at $x > 0$) is free to move. According to Hooke's law, which *defines* the 'ideal spring', the restoring force of the spring is proportional to the displacement, that is,

$$\mathbf{F} = -k(x - L)\hat{\mathbf{x}}$$

where x is the position of the free end and $k > 0$ is the *spring constant*. Glue an object of mass m to the free end, and, for simplicity, shift the labelling of the x axis so that the fixed end of the spring is at $x = -L$. Then the object at the free end is at the origin when the spring is relaxed. The force on the object is

$$\mathbf{F} = -k\hat{\mathbf{x}},$$

and Newton's second law $m\mathbf{a} = \mathbf{F}$ reads

$$m\ddot{x} = -kx.$$

Define

$$\omega \coloneqq \sqrt{\frac{k}{m}}$$

to obtain

$$\ddot{x} + \omega^2 x = 0$$

with general solution

$$x(t) = A \sin \omega t + B \cos \omega t.$$

If we choose the origin of time in such a way that $x(0) = 0$, then this reduces to

$$x(t) = A \sin \omega t.$$

The speed of the object is

$$\dot{x}(t) = \omega A \cos \omega t$$

and the maximum speed, obtained at the origin, is ωA.

1.4.5 The Simple Pendulum

Consider a pendulum consisting of a ball of mass m attached to a string of length L inside a constant gravitational field. Let the angle between the vertical and the string be $\varphi(t)$ at time t. The force on the ball due to gravity is $\mathbf{F}_g = -mg\hat{\mathbf{y}}$. This force can be decomposed into a component $\mathbf{F}_n = mg \cos\varphi \, \hat{\mathbf{n}}$ in the direction of the string (that is, orthogonal to the path of the ball), and a component $\mathbf{F}_\varphi = -mg \sin\varphi \, \hat{\boldsymbol{\varphi}}$ parallel to the path of the ball; see Figure 11. If we consider only small oscillations of the pendulum, say $|\varphi| < \pi/4$, then $\sin x \sim x$ is a decent approximation, and therefore we set

$$\mathbf{F}_\varphi = -mg\varphi\hat{\boldsymbol{\varphi}}.$$

This is the net force acting on the ball, because the force from the string is exactly the opposite of $\mathbf{F}_n$. Hence, Newton's second law states

$$mL\ddot{\varphi} = -mg\varphi$$

where $a \coloneqq L\ddot{\varphi}$ is the acceleration of the ball. Rearranging, we end up with

$$\ddot{\varphi} + \omega^2\varphi = 0$$

where

$$\omega \coloneqq \sqrt{\frac{g}{L}}$$

and the solution is

$$\varphi(t) = A \sin \omega t$$

if we choose the origin of time such that $\varphi(0) = 0$. Recall that this is merely an approximation valid for small oscillations (small A).

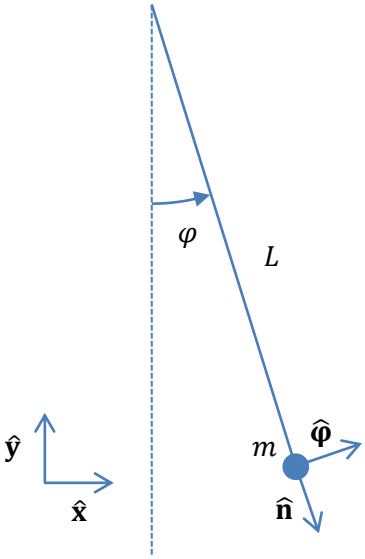


Figure 11. A simple pendulum

1.4.6 The Catenary

We end this section with a proof of the fact that the ideal hanging cable or chain forms the graph of the hyperbolic cosine. This curve is called the *catenary*. Consider so a chain of 'linear density'

ρ [unit: kg/m] hanging between [the tops of] two vertical poles of equal height, forming the image of a parameterisation function $\mathbf{r}(s) = \big(x(s), y(s)\big)$, which we also assume to be a graph of a function $x \mapsto y(x)$. Our aim is to find $x(s)$ and $y(s)$, and then use these formulae to deduce the expression for $y(x)$ by finding $s(x)$. As one might guess, the tricky part is to find the parameterisation. Let us first agree on a coordinate system: Let the $\hat{\mathbf{x}}$ basis vector be parallel with the displacement between the tops of the poles and let $\hat{\mathbf{y}}$ be parallel with each pole, and with the force of gravity. Then put the origin at the point of symmetry of the hanging chain, that is, at its lowest point, situated midway between the poles. In addition, we agree to let $s = 0$ here.

With no loss of generality, we demand that $s \mapsto \mathbf{r}(s)$ be a *unit-speed* parameterisation (as indicated by the usage of the letter 's' for the parameter). This means simply that the parameter is the 'arc-length parameter', that is, the arc length of the curve between parameter values s_1 and $s_2 > s_1$ is exactly $s_2 - s_1$.[7] Then $\hat{\mathbf{t}}(s) \coloneqq d\mathbf{r}(s)/ds$ is the unit tangent vector to the chain at s.

Consider in particular a small segment of the chain, as shown in Figure 12. The small segment is of length Δs and situated between (x, y) and $(x + \Delta x, y + \Delta y)$, corresponding to parameter values s and $s + \Delta s$, respectively. Let the mass of this segment be $\Delta m \coloneqq \rho \Delta s$. This segment is at rest, and so the net force on it must vanish. The net force is the vector sum of three forces, namely,

- the force $\mathbf{F}_g = \Delta m\, \mathbf{g} = -\Delta m\, g \hat{\mathbf{y}}$ of gravity, pointing downwards,
- the force $\mathbf{F}_1 = -\tau(s)\hat{\mathbf{t}}(s)$ from the preceding (smaller s) part of the chain, pointing 'backwards' along the chain, and
- the force $\mathbf{F}_2 = \tau(s + \Delta s)\hat{\mathbf{t}}(s + \Delta s)$ from the following (greater s) part of the chain, pointing 'forwards' along the chain,

where $\tau(s) > 0$ is the *tension* of the chain at s. The assumptions made implicitly above are well motivated. Indeed, $\neg(\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{0})$, for otherwise the net force $\mathbf{F} = \mathbf{F}_g \neq \mathbf{0}$, causing this segment of the chain to accelerate. Also, the existence of the map τ follows from Newton's third law.

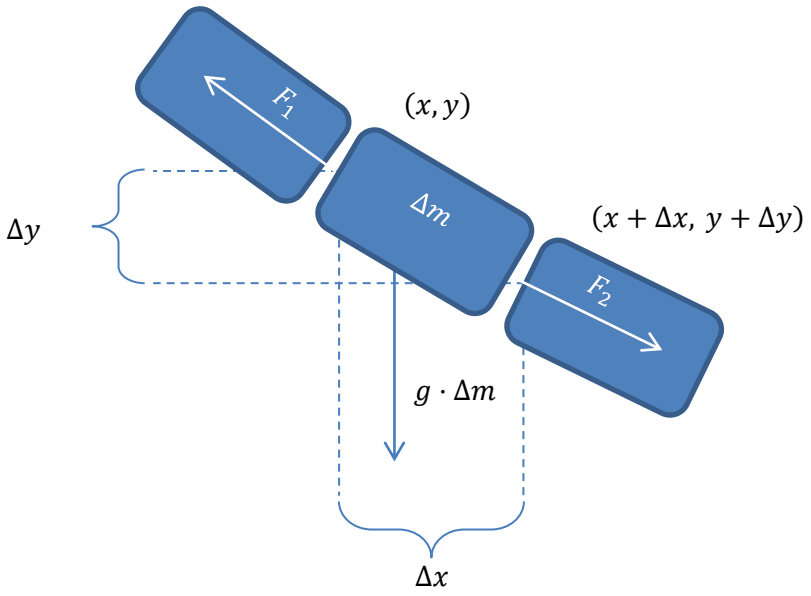


Figure 12. A small segment of a hanging cable or chain.

[7] We will discuss curves in much more detail in the chapter on classical differential geometry.

Now, the observation

$$\mathbf{F}_g + \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$$

reads

$$-\rho g\hat{\mathbf{y}}\Delta s - \tau(s)\hat{\mathbf{t}}(s) + \tau(s+\Delta s)\hat{\mathbf{t}}(s+\Delta s) = \mathbf{0},$$

or, equivalently,

$$\frac{\tau(s+\Delta s)\hat{\mathbf{t}}(s+\Delta s) - \tau(s)\hat{\mathbf{t}}(s)}{\Delta s} = \rho g\hat{\mathbf{y}}.$$

Let $\Delta s \to 0^+$. Then, by the definition of the derivative of a vector-valued function,

$$\frac{d}{ds}[\tau(s)\hat{\mathbf{t}}(s)] = \rho g\hat{\mathbf{y}}$$

yielding

$$\tau(s)\hat{\mathbf{t}}(s) = (a, \rho g s + b)$$

for a pair $(a, b) \in \mathbb{R}^2$ of constants. But since the curve is at its lowest point – and, therefore, is horizontal – at $s = 0$, we have to set $b = 0$ [$\tau(0) \neq 0$ is obvious from the physics of the situation]. Using the definition (↑) of the unit tangent vector, (↑) reads

$$\left(\frac{dx}{ds}, \frac{dy}{ds}\right) = \left(\frac{a}{\tau(s)}, \frac{\rho g s}{\tau(s)}\right).$$

Therefore,

$$\frac{dy}{dx} = \frac{dy}{ds}\cdot\frac{ds}{dx} = \frac{dy}{ds}\cdot\left(\frac{dx}{ds}\right)^{-1} = \frac{\rho g s}{\tau(s)}\cdot\frac{\tau(s)}{a} = \frac{\rho g s}{a} = \lambda s$$

where

$$\lambda \coloneqq \frac{\rho g}{a}$$

is constant. This tells us that the slope of the curve is proportional to the arc length! Unfortunately, we cannot easily integrate (↑) w.r.t. x to obtain the sought expression $y(x)$, so we have to work a bit more. The arc length from the start of the chain to the point (x, y) is

$$s(x) = \int_0^x ds = \int_0^x \sqrt{1 + y'(x')^2}\,dx',$$

whence

$$\frac{ds}{dx} = \sqrt{1 + y'(x)^2}.$$

Of course, this applies to *any* curve $y = y(x)$ with arc length $s(x)$. But in *this* case, (↑) can be used to yield

$$\frac{ds}{dx} = \sqrt{1 + \lambda^2 s^2}.$$

Therefore,

$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + \lambda^2 s^2}}.$$

This can be integrated with respect to s:

$$x(s) = \frac{1}{\lambda}\operatorname{arcsinh}\lambda s$$

respecting the condition $x(0) = 0$. Having found an expression for the *first* parameterisation function $s \mapsto x(s)$, we turn to the *second* function parameterising the curve, i.e., $s \mapsto y(s)$. This is actually rather simple, for

$$\frac{dy}{ds} = \frac{dy}{dx}\cdot\frac{dx}{ds} = \frac{\lambda s}{\sqrt{1+\lambda^2 s^2}}$$

using (↑) and (↑) and so

$$y(s) = \frac{1}{\lambda}\sqrt{1+\lambda^2 s^2} - \frac{1}{\lambda}$$

respecting $y(0) = 0$. We have thus found our parameterisation $s \mapsto \mathbf{r}(s)$ – isn't that exciting! Following our recipe, (↑) is solved with respect to s, yielding

$$s(x) = \frac{1}{\lambda}\sinh\lambda x$$

which is inserted into (↑) to yield

$$y(x) = \frac{1}{\lambda}\sqrt{1+\sinh^2\lambda x} - \frac{1}{\lambda} = \\ = \frac{1}{\lambda}(\cosh\lambda x - 1)$$

using the hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$ and recalling that the hyperbolic cosine is a positive function[8]. What is the meaning of λ? Well, if the [tops of the] poles are positioned at $(\pm d, h)$, then it is required that

$$h = \frac{1}{\lambda}(\cosh\lambda d - 1).$$

which is an implicit equation for λ as a function of d and h. The result that a hanging chain forms the graph of the hyperbolic cosine function can be verified 'numerically', by trying to fit different curves to photographs of actual real-world chains. See Figure 13.

[8] Naturally, with foresight, we could have put the origin a distance $1/\lambda$ below the chain's point of symmetry, and then we would have ended up with the slightly cuter formula $y(x) = \frac{1}{\lambda}\cosh\lambda x$.

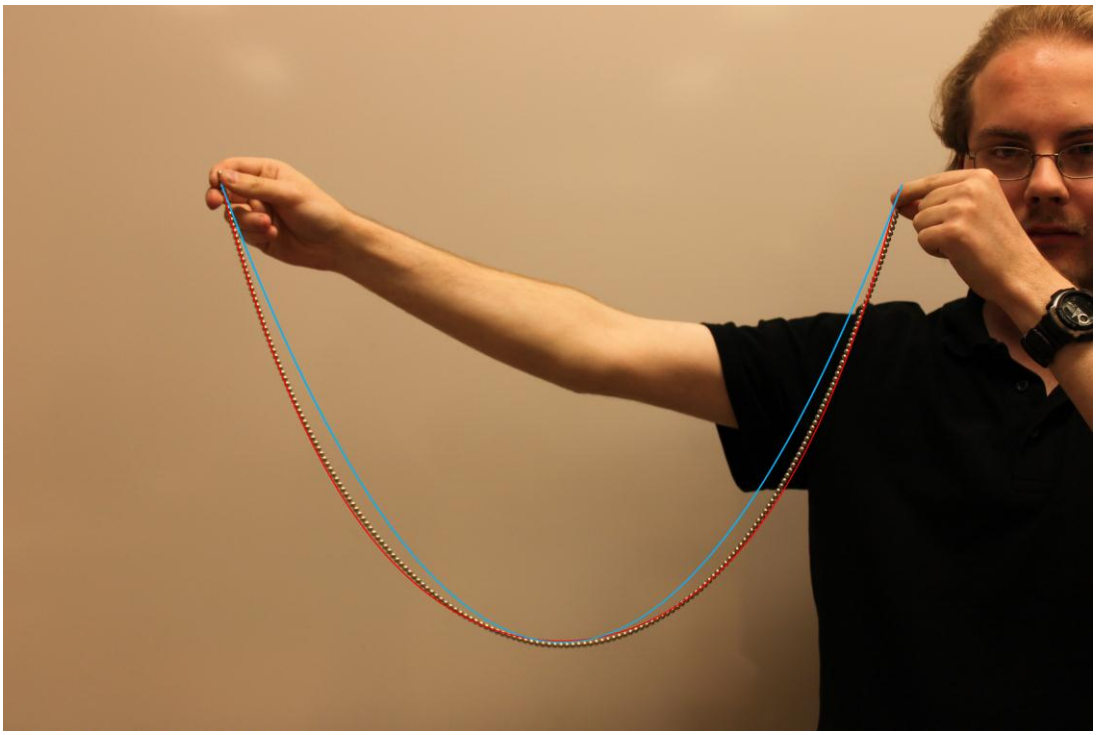


Figure 13. A hanging chain and a superimposed parabola (blue) and catenary (red).

1.5 Momentum and Collisions

1.5.1 Momentum

The usefulness of the Newtonian concept of 'momentum' – defined as $\sum m_i \mathbf{u}_i$ for an isolated system of particles with masses m_i and velocities $\mathbf{u}_i$ – lies in the fact that the momentum so defined is a conserved quantity when computed in any inertial frame. This follows immediately from Newton's laws. For example, let A and B be two billiard balls (or, more generally, particles) in empty space, with masses m_A and m_B and velocities $\mathbf{u}_A$ and $\mathbf{u}_B$, respectively. If they do not collide or interact via long-range forces, then the total momentum $\sum m_i \mathbf{u}_i$ is conserved in time, as dictated by Newton's first law. Therefore, let us assume that they do collide or interact via long-range forces. Then, let $\mathbf{F}_{AB}(t)$ be the force *on* A *due to* B, and let $\mathbf{F}_{BA}(t)$ be the force on B *due to* A, at time t. Newton's third law requires $\mathbf{F}_{AB}(t) \equiv -\mathbf{F}_{BA}(t)$. Thus, using the second law, we have

$$\frac{d}{dt}(m_A \mathbf{u}_A) = -\frac{d}{dt}(m_B \mathbf{u}_B),$$

or, equivalently,

$$\frac{d}{dt}(m_A \mathbf{u}_A + m_B \mathbf{u}_B) = 0.$$

That is, total momentum is a constant in time. This argument is readily generalized to $n > 2$ particles, and to a continuum ($n \to \infty$) of matter; we will do so in the next section. Notice in particular that, although the momentum of an isolated system is different as seen from different physical frames, the conservation, or constancy in time, of momentum holds equally well in any inertial frame. For example, let two balls (A and B) of equal masses m approach each other along the x axis of $\mathcal{F}1$, about to collide completely elastically at the origin. Let them have velocities $5\hat{\mathbf{x}}$ and $-5\hat{\mathbf{x}}$ prior to the collision, and velocities $-5\hat{\mathbf{x}}$ and $5\hat{\mathbf{x}}$ after the collision. Now, let $\mathcal{F}2$ be a different frame in standard configuration relative to $\mathcal{F}1$, where the origin of $\mathcal{F}2$ has velocity $3\hat{\mathbf{x}}$ relative to $\mathcal{F}1$. As seen from $\mathcal{F}2$, the pre-collision velocities are $2\hat{\mathbf{x}}$ and $-8\hat{\mathbf{x}}$. The post-collision velocities are $-8\hat{\mathbf{x}}$ and $2\hat{\mathbf{x}}$. Hence, as seen from the point of view of $\mathcal{F}1$, the total momentum is 'changed' form $0\hat{\mathbf{x}}$ to $0\hat{\mathbf{x}}$, and from the point of view of $\mathcal{F}2$, the total momentum is 'changed' from $-6m\hat{\mathbf{x}}$ to $-6m\hat{\mathbf{x}}$. A third example: Let $\mathcal{F}3$ be the pre-collision rest frame of B. In this frame, the initial velocities are $10\hat{\mathbf{x}}$ and $0\hat{\mathbf{x}}$, the final velocities are $0\hat{\mathbf{x}}$ and $10\hat{\mathbf{x}}$ and so the momentum 'changes' from $10m\hat{\mathbf{x}}$ to $10m\hat{\mathbf{x}}$. [Notice that $\mathcal{F}3$ also serves as the *post*-collision rest frame of A.] We will investigate collisions that are (seemingly!) more general after the next section.

1.5.2 Many-Particle Systems

Now consider an isolated system of N particles in space and let

$$X \coloneqq \{1, 2, \dots, N\}$$

be the set of all particle indices; from now on, we will always refer to this set as the 'index set' of the system. Assume that the mass of particle i is m_i and that it is located at $\mathbf{r}_i$. The total momentum is

$$\mathbf{p} = \sum_{i \in X} \mathbf{p}_i = \sum_{i \in X} m_i \mathbf{u}_i$$

where $\mathbf{p}_i \coloneqq m_i \mathbf{u}_i$ is the momentum of the ith particle. Now consider the most general case, in which the force *on* the ith particle *due to* the jth particle at time t is $\mathbf{F}_{ij}(t)$. Of course, a particle

does not affect itself by any non-zero force, so *a priori* there are no symbols of the form $\mathbf{F}_{ii}$; for notational simplicity, however, we define

$$\mathbf{F}_{ii}(t) \coloneqq \mathbf{0}, \qquad \forall i \in X.$$

Then Newton's second law, as applied to particle i, can be written very succinctly

$$m_i \mathbf{a}_i(t) = \sum_{j \in X} \mathbf{F}_{ij}(t)$$

while Newton's third law ensures that

$$\mathbf{F}_{ij}(t) \equiv -\mathbf{F}_{ji}(t), \qquad \forall (i,j) \in X^2.$$

It follows immediately from Newton's laws that the total momentum $\mathbf{p}$ is a constant of motion.[9] Indeed, for every force $\mathbf{F}_{ij}$ that changes the momentum of the ith particle by an amount $\frac{d}{dt}(m_i \mathbf{u}_i) = \mathbf{F}_{ij}$, the momentum of the jth particle is changed by an opposite amount $\frac{d}{dt}(m_j \mathbf{u}_j) = \mathbf{F}_{ji} = -\mathbf{F}_{ij}$. More formally,

$$\frac{d}{dt}\mathbf{p} = \frac{d}{dt}\sum_{i \in X} m_i \mathbf{u}_i = \sum_{i \in X} \frac{d}{dt}(m_i \mathbf{u}_i) = \sum_{i \in X}\left(\sum_{j \in X} \mathbf{F}_{ij}\right) = \sum_{(i,j) \in X^2} \mathbf{F}_{ij} = \mathbf{0}$$

since the vector-valued matrix $\mathbf{F}_{ij}$ is skew. We have thus shown

Theorem

The total momentum of an isolated system of discrete material particles is constant in time.

We now define the *centre of mass* $\mathbf{r}_{\text{cm}}$ as the 'weighted' average position with respect to the particle masses, that is,

$$\mathbf{r}_{\text{cm}} \coloneqq \frac{1}{M}\sum_{i \in X} m_i \mathbf{r}_i$$

where m_i is the mass of the ith particle and

$$M \coloneqq \sum_{i \in X} m_i$$

is the total mass of the system. The acceleration of the centre of mass is

$$\mathbf{a}_{\text{cm}} \coloneqq \frac{d^2}{dt^2}\left(\frac{1}{M}\sum_{i \in X} m_i \mathbf{r}_i\right) = \frac{1}{M}\frac{d}{dt}\left(\sum_{i \in X} \frac{d}{dt}(m_i \mathbf{r}_i)\right) = \frac{1}{M}\frac{d}{dt}\left(\sum_{i \in X} m_i \mathbf{u}_i\right) = \frac{1}{M}\frac{d}{dt}\mathbf{p} = \mathbf{0}$$

and therefore we can use the centre of mass as the origin of an inertial frame. This frame (or, rather, such a frame) is called the *centre of mass frame*, or the 'CM frame'.

Proposition

As seen from the CM frame, the total momentum of a system of matter particles is zero.

[9] In physical jargon, a 'constant of motion' is a quantity that doesn't change in time.

Proof

Let $\mathbf{v}_i$ be the velocity of the ith particle relative to the ambient reference frame, and let $\mathbf{u}_i$ be the velocity of the same particle relative to the CM frame, so that $\mathbf{v}_i = \mathbf{v}_{\text{cm}} + \mathbf{u}_i$. Then we want to show that $\sum_{i\in X} m_i\mathbf{u}_i = \mathbf{0}$. And, indeed,

$$\sum_{i\in X} m_i\mathbf{u}_i = \sum_{i\in X} m_i(\mathbf{v}_i - \mathbf{v}_{\text{cm}}) = \sum_{i\in X} m_i\mathbf{v}_i - \sum_{i\in X} m_i\mathbf{v}_{\text{cm}} = \frac{d}{dt}\sum_{i\in X} m_i\mathbf{r}_i - \mathbf{v}_{\text{cm}}\sum_{i\in X} m_i =$$
$$= \frac{d}{dt}(M\mathbf{r}_{\text{cm}}) - M\mathbf{v}_{\text{cm}} = M\mathbf{v}_{\text{cm}} - M\mathbf{v}_{\text{cm}} = \mathbf{0}.$$

■

Proposition

Let $\mathcal{F}1$ be an inertial frame relative to which the centre of mass of a system of matter particles moves with velocity $\mathbf{v}_{\text{cm}}$. If the system has total mass M, then, as seen from $\mathcal{F}1$, the total momentum of the system is $\mathbf{p} = M\mathbf{v}_{\text{cm}}$.

Proof

Let $\mathbf{v}_i$ be the velocity of the ith particle relative to $\mathcal{F}1$ and $\mathbf{u}_i$ be the velocity of the same particle relative to the CM frame, so that $\mathbf{v}_i = \mathbf{v}_{\text{cm}} + \mathbf{u}_i$. Then, the total momentum of the system as seen from $\mathcal{F}1$ is

$$\mathbf{p} = \sum_{i\in X}\mathbf{p}_i = \sum_{i\in X} m_i\mathbf{v}_i = \sum_{i\in X} m_i(\mathbf{v}_{\text{cm}} + \mathbf{u}_i) = \sum_{i\in X} m_i\mathbf{v}_{\text{cm}} + \sum_{i\in X} m_i\mathbf{u}_i = \mathbf{v}_{\text{cm}}\sum_{i\in X} m_i = M\mathbf{v}_{\text{cm}}$$

because $M \overset{\text{def}}{=} \sum_{i\in X} m_i$ and $\sum_{i\in X} m_i\mathbf{u}_i = \mathbf{0}$ according to Proposition NN. ■

Proposition

Consider a system of particles with total mass M. Assume that the ith particle is affected by an *external* force $\mathbf{F}_i$ in addition to the internal forces $\mathbf{F}_{ij}$ from the other particles inside the system. Then

$$\sum_{i\in X}\mathbf{F}_i = M\mathbf{a}_{\text{cm}}$$

where $\mathbf{a}_{\text{cm}}$ is the acceleration of the centre of mass of the system.

Proof

Now Newton's second law on the ith particle reads

$$m_i\mathbf{a}_i = \sum_{j\in X}\mathbf{F}_{ij} + \mathbf{F}_i$$

and so, using the definition of the centre of mass $\mathbf{r}_{\text{cm}}$ and the fact that $\mathbf{F}_{ij}$ is skew,

$$M\mathbf{a}_{\text{cm}} = \sum_{i\in X} m_i\mathbf{a}_i = \sum_{i\in X}\left(\sum_{j\in X}\mathbf{F}_{ij} + \mathbf{F}_i\right) = \sum_{(i,j)\in X^2}\mathbf{F}_{ij} + \sum_{i\in X}\mathbf{F}_i = \sum_{i\in X}\mathbf{F}_i.$$

■

The main point of the last three propositions is that, a system of material particles can be thought of as a single particle of mass M located at $\mathbf{r}_{\text{cm}}$ as long as the internal structure of the system is not of any interest.

1.5.3 Momentum Conservation and the Galilean Transformation

Assume that we have found (either theoretically or empirically) that the total momentum of an isolated system is conserved in one inertial frame $\mathcal{F}1 \in \mathcal{F}$. It is then natural to ask whether momentum is seen to be conserved in another inertial frame $\mathcal{F}2 \in \mathcal{F}$, as well. The answer is yes, because we have seen that the law of momentum conservation follows immediately from Newton's laws of motion, and we have postulated/shown (depending on how you treat the concept of 'force') that these laws are equally valid in any inertial frame. However, it is instructive to given an alternative – and far more convincing – proof of this fact without resorting to the complicated matter of the concept of 'force'. We will do so now.

Proposition

Consider a system of N particles with index set X. Let the total momentum be $\mathbf{p} \coloneqq \sum_{i\in X} m_i\mathbf{u}_i$ as seen from an inertial frame $\mathcal{F}1 \in \mathcal{F}$ and let $\mathbf{p}' \coloneqq \sum_{i\in X} m_i\mathbf{u}_i'$ be the total momentum as seen from a different inertial frame $\mathcal{F}2 \in \mathcal{F}$. Then

$$\frac{d}{dt}\mathbf{p} = \mathbf{0} \Rightarrow \frac{d}{dt}\mathbf{p}' = \mathbf{0}.$$

Proof

If $\mathbf{x}$ and $\mathbf{x}'$ are the locations of some particle relative to $\mathcal{F}1$ and $\mathcal{F}2$, respectively, we have

$$\mathbf{x} = \mathbf{x}' + \mathbf{v}t + \boldsymbol{\delta}$$

where $\mathbf{v}$ is the (constant) velocity of the origin of $\mathcal{F}2$ relative to $\mathcal{F}1$ and $\boldsymbol{\delta}$ is the displacement from the origin of $\mathcal{F}1$ to the origin of $\mathcal{F}2$ at the origin of time. Differentiation yields ($\dot{\mathbf{x}} = \mathbf{u}$)

$$\mathbf{u} = \mathbf{u}' + \mathbf{v}.$$

Thus

$$\frac{d}{dt}\mathbf{p}' = \frac{d}{dt}\left(\sum_{i\in X} m_i\mathbf{u}_i'\right) = \frac{d}{dt}\left(\sum_{i\in X} m_i(\mathbf{u}_i - \mathbf{v})\right) = \frac{d}{dt}\left(\sum_{i\in X} m_i\mathbf{u}_i\right) - \left(\sum_{i\in X} m_i\frac{d\mathbf{v}}{dt}\right) = \frac{d}{dt}\mathbf{p}$$

since $d\mathbf{v}/dt = 0$ and the proposition follows. ■

1.5.4 General Collisions

We now return to our discussion of collisions. Let A travel along the x axis of $\mathcal{F}1 \in \mathcal{F}$ with velocity $5\hat{\mathbf{x}}$ and let B travel along the y axis with velocity $5\hat{\mathbf{y}}$. Let them approach the origin from minus infinity [along the respective axes], and let them collide at the origin. After the collision, A will have velocity $5\hat{\mathbf{y}}$ and B will have velocity $5\hat{\mathbf{x}}$. Thus, the momentum is 'changed' from $5m\hat{\mathbf{x}} + 5m\hat{\mathbf{y}}$ to $5m\hat{\mathbf{x}} + 5m\hat{\mathbf{y}}$. At a first glance, one might think that this collision is fundamentally different from the three previous examples. Indeed, the earlier examples were 'head-on' collisions, in which the velocity vectors were parallel both before and after the collision, whereas the velocity vectors are perpendicular in this last example. This, however, *is not a (geometric) property of the collision*, but depends on the frame of reference! To see this, let $\mathcal{F}2 \in \mathcal{F}$ be the rest frame of B; notice that $\mathcal{F}1 \nsim \mathcal{F}2$. The transformation between $\mathcal{F}1$ and $\mathcal{F}2$ is a Galilean transformation, which

is linear, and so even in this frame, A travels along a straight line, still with constant speed. Thus, in $\mathcal{F}2$, A is seen to approach the stationary second ball B along a straight line with constant speed, and this is a typical example of a 'head-on' collision as discussed previously.

We will now describe an alternative – and perhaps more intuitive – way of seeing this. At any pre-collision time t, let $\ell(t)$ be the straight line that passes through A and B. Obviously, every $\ell(t)$ has the normal direction $\hat{\mathbf{x}}+\hat{\mathbf{y}}$ and $\ell(t)$ is being parallel propagated in this direction as time passes. Further, let $P(t)\in\ell(t)$ be the centre of mass of the system (A,B) at time t, that is, the *midpoint* of the segment of $\ell(t)$ between A and B at this time. It follows that the velocity $\dot{P}(t)$ of this point is parallel to $\hat{\mathbf{x}}+\hat{\mathbf{y}}$ and constant in time; also, the speed $|\dot{P}(t)|$ is equal to the 'speed of propagation' of $\ell(t)$. Denote this speed by $v_\ell := |\dot{P}(t)|$. The idea is to introduce a new frame $\mathcal{F}2\in\mathcal{F}$ the origin of which is $P(t)$, and the first basis vector $\hat{\mathbf{x}}'$ of which is parallel to $\ell(t)$ [this direction being independent of time], pointing from A to B. In this frame, the two balls are approaching each other along the x' axis, i.e., the velocities are $u\hat{\mathbf{x}}'$ and $-u\hat{\mathbf{x}}'$, respectively, for some constant $u>0$ that the reader can easily determine for herself should she feel the need to do so. More importantly, this is *the* typical 'head-on' situation.

Since the components of a vector, such a velocity vector, do not depend on the actual position of the origin, we can simplify the recipe used above to obtain a frame $\mathcal{F}2$ in which the collision is 'head-on' with velocities $u\hat{\mathbf{x}}'$ and $-u\hat{\mathbf{x}}'$. Start with the frame $\mathcal{F}1\in\mathcal{F}$, and introduce a new mathematical frame $\mathcal{F}1'\in\mathcal{F}$, $\mathcal{F}1'\sim\mathcal{F}1$ that is rotated 45° clockwise. Then introduce a new frame $\mathcal{F}2\in\mathcal{F}$ in standard configuration along the y axis relative to $\mathcal{F}1'$ with relative speed $v=v_\ell$. As seen from $\mathcal{F}2$, the pre-collision velocities are $u\hat{\mathbf{x}}'$ and $-u\hat{\mathbf{x}}'$.

We have thus seen that, if two balls of equal mass and speed are found to collide with their velocities perpendicular to each other, there is always another frame in which they collide with their velocities parallel. Conversely, if two such balls are found to collide with their velocities parallel, we can find a frame in which they collide with their velocities perpendicular to each other. All we have to do is to 'reverse' the argument given above. Indeed, in $\mathcal{F}2$ the collision is 'head on', but if we move the observer to $\mathcal{F}1$, the pre-collision velocities are perpendicular to each other.

We will now treat the most general case of a two-particle collision. Let $\mathcal{F}1\in\mathcal{F}$ be a frame in which A and B are two balls with masses m_A and m_B and velocities $\mathbf{u}_A$ and $\mathbf{u}_B$. The speeds u_A and u_B need not be equal, and the angle between the velocities can be any number, with one single restriction: they have to collide with each other at some time in the future (obviously!). We will show that there exists an inertial frame $\mathcal{F}2\in\mathcal{F}$ in which the pre-collision velocities are $u\hat{\mathbf{x}}'$ and $-u\hat{\mathbf{x}}'$ for some $u>0$. In fact, we have already found a hint about how to do this: *introduce the centre of mass frame*. This is an inertial frame, as shown above, and in this frame, the total momentum $\mathbf{p}'$ of the two balls is zero, that is,

$$\mathbf{p}'=m_A\mathbf{u}_A'+m_B\mathbf{u}_B'=\mathbf{0}$$

where $\mathbf{u}_A'$ and $\mathbf{u}_B'$ are the velocities of the balls as seen from this frame. It follows that

$$\mathbf{u}_A'=-\frac{m_B}{m_A}\mathbf{u}_B',\qquad\text{i.e.}\qquad\mathbf{u}_A'\parallel\mathbf{u}_B',$$

and so this is the sought frame (just pick a basis parallel with $\mathbf{u}_A'$). Perhaps the most important morale of the story is

Observation N

The angle between two velocity vectors depends upon the inertial frame of reference. In other words, the concept 'angle between two velocity vectors' lacks intrinsic meaning.

When applied to collisions, we first make

Definition N + 1

A collision between two free particles is said to be *head on* iff there exists an inertial frame relative to which the pre-collision velocity vectors are parallel.

in order to formulate

Observation N + 2

Every collision between two free particles is head on.

The requirement that the particles be free does not involve any deep insight. It is only there because, if it were not, we could not as easily talk about the 'pre-collision velocities'. Indeed, if one of the particles is not free, then it is affected by forces, and so its velocity changes in time. Hence, there is no single 'pre-collision velocity'.

Combining the above results with Proposition NN, we have

Observation N + 3

Consider a collision between two free particles. In the CM frame, the pre-collision velocities are parallel and opposite, and the velocities change direction at the collision.

1.6 Energy

The concept of 'energy' is one of the cornerstones of modern physics, and during the last century or so, it has also become of widespread interest in every-day life, although in a less technical language. Indeed, the modern society exploits energy sources to such an extent that the entire environment of the Earth is endangered. In this section, we will investigate the physical basis of the energy concept.

1.6.1 Work and Kinetic Energy

The basis of the entire concept of 'energy' is

Definition

Let a particle have mass m and speed u. The *kinetic energy* of the particle is defined as

$$E_k = \frac{1}{2}mu^2.$$

At first sight, this looks just like a different measure of the speed of an object, and – indeed – we have no *a priori* reason to believe that this quantity is of any particular interest. The concept used to motivate this definition is the concept of *work*:

Definition

Assume that the net force field[10] in some region $D \subseteq \mathbb{R}^3$ in space is $\mathbf{F}(\mathbf{x})$ where $\mathbf{x} = (x, y, z) \in D$. Assume that the particle, which lives inside the region D and is only affected by the force field $\mathbf{F}$, follows the curve $\Gamma \subset D$. In general, Γ will not be a straight line. The *work* done on the particle by the force field is defined as the line integral

$$W = \int_\Gamma \mathbf{F}(\mathbf{x}) \cdot d\mathbf{l}.$$

Kinetic energy and work are very closely related concepts; indeed, the latter is the change in the former. More precisely, assume that the particle travels between points P and Q in space along a curve Γ, under the influence only of the force field $\mathbf{F}$. Let $E_P = \frac{1}{2}mu_P^2$ and $E_Q = \frac{1}{2}mu_Q^2$ be the kinetic energy at P and Q, respectively, where the instantaneous speed of the particle is u_P and u_Q (also respectively). Let $\Delta E_k \coloneqq E_Q - E_P$ be the gain in kinetic energy during the journey (possibly zero or negative), and let W be the work done on the particle by the force field. Then we have

Theorem (The Work—Energy Theorem)

$$\Delta E_k = W.$$

[10] *In this section*, a 'force field' $\mathbf{F}$ is a vector field in space such that *a particular* body located at $\mathbf{x}$ experiences the force $\mathbf{F}(\mathbf{x})$. In other words, the field depends on the particular body of consideration. For instance, consider the gravitational field from a star. A heavy planet located at some point P would experience a greater force than a smaller planet located at the very same point P. This means that, *in this section*, when we talk about the 'force field' from the star, we mean the field experienced by some pre-chosen test body. In later sections, we will employ the more natural definitions of force fields as the fields experienced by a particle of *unit* mass (in the case of a gravitational field) or *unit* charge (in the case of an electrostatic field). *In this section*, however, it is more illustrative to talk about the force field that is the actual force on an actual body.

Proof

Let Γ be parameterised by $\mathbf{x}\colon I \to D$ where $I = [0,1]$ is the unit interval; consequently, $\mathbf{x}(0) = P$ and $\mathbf{x}(1) = Q$. Then

$$W \overset{\text{def}}{=} \int_\Gamma \mathbf{F}(\mathbf{x}) \cdot d\mathbf{l} = \int_0^1 \mathbf{F}\big(\mathbf{x}(t)\big) \cdot \dot{\mathbf{x}}(t) dt = \int_0^1 m\ddot{\mathbf{x}}(t) \cdot \dot{\mathbf{x}}(t) dt = m \int_0^1 \frac{d}{dt}\left(\frac{1}{2}\dot{\mathbf{x}}(t)^2\right) dt =$$
$$= \frac{1}{2} m\big(u_Q^2 - u_P^2\big) \overset{\text{def}}{=} \Delta E_k$$

where we took the liberty of using Newton's second law. ■

Corollary

Let a particle travel along a curve Γ in a (net) force field $\mathbf{F}$ such that $\mathbf{F}(\mathbf{x})$ is orthogonal to Γ at every point $\mathbf{x} \in \Gamma$. Then the work $W = 0$ and so the kinetic energy of the particle remains constant during the journey along Γ.

1.6.2 Conservative Forces

Many force fields of physical interest, such that the force fields of gravity and electrostatics, are *conservative*; a vector field $\mathbf{F}$ is said to be 'conservative' in a domain D iff there exists a scalar field ϕ such that

$$\mathbf{F}(\mathbf{x}) = -\nabla\phi(\mathbf{x}), \qquad \forall \mathbf{x} \in D.$$

The scalar field ϕ is called the 'potential' of the vector field, and is determined by $\mathbf{F}$ up to an additive constant.[11] A conservative force field has the interesting property that the line integral between two points P and Q is independent of the actual path Γ we integrate along, as long as – of course – the path starts at P and ends at Q. In fact, the line integral of $\mathbf{F}$ along any path Γ starting at P and ending at Q is equal to the negative[12] of the potential difference $\phi(Q) - \phi(P)$, that is,

$$W = -\big(\phi(Q) - \phi(P)\big).$$

Path independence clearly implies that the curve integral along any *closed* curve Γ vanishes, since $P = Q \Rightarrow \phi(Q) - \phi(P) = 0$. The converse is also true, as is easily deduced. Moreover, path independence between every pair of points implies that the field is conservative. We also notice that the vector identity $\nabla \times (\nabla\phi) \equiv \mathbf{0}$ implies that every conservative vector field is irrotational; the converse is only true if the domain is simply connected, however.

The gravitational field and the electrostatic field are both conservative. This means, for instance, that the speed (or, equivalently, kinetic energy) of a planet orbiting a star in a closed elliptic orbit is the same each time the planet occupies the same point in the orbit. More generally, consider *any* body in the vicinity of a star, and assume that it is only affected by the gravitational field from the star. If we know the speed of the body at *some* point, we can deduce its speed at *any*

[11] In pure mathematical texts the potential is often defined by $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$ instead, without the minus sign, and a vector field is said to be conservative if it has a potential ϕ such that $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$. Although this definition yields another set of admissible potentials, the definitions of 'conservative' agree. Indeed, $\exists\phi_1\colon \mathbf{F}(\mathbf{x}) = \nabla\phi_1(\mathbf{x}) \Leftrightarrow \exists\phi_2\colon \mathbf{F}(\mathbf{x}) = -\nabla\phi_2(\mathbf{x})$; for example, you can always choose $\phi_1(\mathbf{x}) \equiv -\phi_2(\mathbf{x})$.

[12] With the mathematician's convention $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$ we would have been relieved from the minus sign. That is, we would have $\int_\Gamma \mathbf{F}(\mathbf{x}) \cdot d\mathbf{l} = \phi(Q) - \phi(P)$ where $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$. In the one-dimensional case (by which I mean on $\mathbb{R}$), this reduces to $\int_a^b F(x)dx = \phi(b) - \phi(a)$ where $F(x) = \frac{d}{dx}\phi(x)$ [after trivial identification of real numbers and vectors in $\mathbb{R}^1$]. Looks familiar?

other point. In fact, since the potential only depends on the radial distance from the star to the body, it suffices to know the radial distances at both instances.

Now, let us return to the planet orbiting the star in an elliptic orbit. We found that the speed (or, equivalently, kinetic energy) is the same every time the planet visits the same point in the orbit. Let P be a point in the orbit with kinetic energy E_P, and let Q be a point with kinetic energy E_Q. Let Γ be the part of the elliptic orbit between P (start) and Q (end), and let W be the work done on the body as it travels from P to Q along Γ. Then $\Delta E_K \overset{\text{def}}{=} E_Q - E_P = W = -\big(\phi(Q) - \phi(P)\big)$. If $W < 0$, kinetic energy is lost ($E_Q < E_P$) during the journey from P to Q. This is clearly equivalent to $\phi(Q) > \phi(P)$, that is, the *potential* is higher at Q. On the other hand, if $W > 0$, the particle has gained kinetic energy ($E_Q > E_P$), which is equivalent to $\phi(Q) < \phi(P)$, that is, the *potential* is lower at Q. A high potential can thus be seen, loosely speaking, as a 'promise' that the force field can give the planet some additional kinetic energy; in slightly other words, a high potential means that there is a high 'potential' for the kinetic energy to grow[13]. Notice in particular that the quantity

$$E_k + \phi = \frac{1}{2}mu^2 + \phi(\mathbf{x})$$

is a *constant* during the motion, where $\mathbf{x}$ is the current position of the planet. Indeed, fix *some* point P in the orbit. At this point, the kinetic energy is E_P and the potential is $\phi(P)$. Let Q be *any* later point in the orbit. Here the kinetic energy is E_Q and the potential is $\phi(Q)$. But

$$\Delta E \overset{\text{def}}{=} E_Q - E_P = W = -\big(\phi(Q) - \phi(P)\big) \Rightarrow E_Q + \phi(Q) = E_P + \phi(P)$$

which proves the statement. If a particle is located at a point P, then

$$U \coloneqq \phi(P)$$

is called the *potential energy* of the particle. The sum

$$E_m \coloneqq E_k + U$$

of the kinetic and potential energy of a particle (at some point P) is called the *total mechanical energy* of the particle. We have thus shown that the total mechanical energy is *constant* in any conservative force field.

1.6.3 Examples of Force Fields

1.6.3.1 The Gravitational Field

As introduced above, Newton's law of universal gravitation gives the force of gravity on a massive body B, the test particle (mass m_B) due to another massive body A, the source of the field (mass m_A) as

$$\mathbf{F} = -G\frac{m_A m_B}{r^2}\hat{\mathbf{r}}.$$

Choose a spherical coordinate system such that the source A of the field is located at the origin. Then r, the distance from A to B, is equal to the radial coordinate of B, and $\hat{\mathbf{r}}$ is the radial unit vector at B. Since the force is proportional to the mass of the second body, it is convenient to define the gravitational (force) field due to the source A as the force experienced by a *unit-mass* test particle. That is, we define the gravitational field to be

[13] This is the reason why, in physics, we define the potential of a vector field *with* the minus sign.

$$\mathbf{G}(\mathbf{x}) \coloneqq -G\frac{m_A}{r^2}\hat{\mathbf{r}},$$

which now is a property only of the source. Now, *any* massive body B with mass m_B at $\mathbf{x}$ is affected by the force

$$\mathbf{F} = m_B\mathbf{G}(\mathbf{x}).$$

The gravitational potential scalar field ϕ is defined by $\mathbf{G}(\mathbf{x}) = -\nabla\phi(\mathbf{x})$. In polar coordinates,

$$\phi(r,\theta,\varphi) = -G\frac{m_A}{r}.$$

Therefore, the *potential energy* of a test body B with mass m_B located at $\mathbf{x}$ is

$$U = m_B\phi(\mathbf{x}).$$

1.6.3.2 The Gravitational Field at the Surface of the Earth

Locally[14] at the surface of the Earth, the distance r between the Earth's centre of mass and a test body is virtually constant, as is the direction $\hat{\mathbf{r}}$. Thus, we define

$$g \coloneqq \frac{GM}{r^2},$$

where M is the mass of the Earth, to obtain the excellent approximation

$$\mathbf{G}(\mathbf{x}) = -g\hat{\mathbf{y}}$$

(independent of $\mathbf{x}$) where $\hat{\mathbf{y}}$ is a unit vector pointing along the y axis, which we choose to be directed upwards from the ground. Let B be a body with mass m. The force of gravity on B is thus

$$\mathbf{F} = m\mathbf{G} = -mg\hat{\mathbf{y}}.$$

The magnitude $F = mg$ is a familiar expression to everyone. The minus sign simply tells us that the force is directed towards the ground. The potential is

$$\phi(y) = gy$$

because $-\nabla\phi(y) = -g\hat{\mathbf{y}} = \mathbf{G}$. Thus, the potential energy of B is

$$U = mgy.$$

Since the potential is determined by the force field only up to an additive constant, clearly we can choose the origin of y arbitrarily. Indeed, only *differences* in potential energy ever determine a change in kinetic energy.

Example N

At the ground of the Earth,

$$g \approx 9.8\,\mathrm{N/kg} = 9.8\,\mathrm{m/s^2}.$$

Choose the zero of the y axis to be at the ground; thus, the potential energy is zero here. Then, at a height $y = h \coloneqq 3.0$ m, a body of mass $m = 2.0$ kg has potential energy $U = mgh = 59$ J. Let the body be at rest, so its kinetic energy is $E_k = 0$ J. Hence, the total mechanical energy $E_t = E_k + U = 59$ J. Then let go of it. During the fall, potential energy will be 'converted' into kinetic energy. At the time the body hits the ground, the potential energy is $U = 0$ J so the kinetic energy is

[14] In a room in a building, say.

$$E_k = E_t - U = mgh = 59 \text{ J}.$$

But since $E_k \overset{\text{def}}{=} \frac{1}{2}mu^2$ we can easily solve for the speed of impact:

$$u = \sqrt{2E_k/m} = \sqrt{2gh} = 7.7 \text{ m/s}.$$

[Compare this equation with Eq NN obtained from pure kinematic results in Section 1.4.2.] Using other words, during the fall gravity does *work* on the ball. The work is

$$\Delta E_k = W = -m\big(\phi(0) - \phi(h)\big) = mhg.$$

That is, the kinetic energy increases from 0 J by an amount mhg.

Notice once again that the mass of an object does not influence its motion in a gravitational field. In fact, a body of mass m is affected by the force $F = mg$, so that Newton's second law reads

$$ma = mg.$$

Since the passive gravitational mass m in the RHS is equal to the inertial mass m in the LHS, these cancel, producing

$$a = g.$$

That is, g is the acceleration of *any* massive body in free fall near the Earth's surface.

For future reference, we give

Proposition

Consider a system of N massive particles in a constant gravitational field. Then the gravitational potential energy of the system is equal to the potential energy of a single particle of the same mass as the entire system located at the centre of mass of the system.

Proof

Let there be N particles with 'index set' X. Let the mass and position of the ith particle be m_i and $\mathbf{r}_i$, respectively. Assume the force field is $\mathbf{G}(\mathbf{x}) = -g\hat{\mathbf{z}}$ for some $g \in \mathbb{R}$. Then the total potential energy is

$$U = \sum_{i\in X} U_i = \sum_{i\in X} m_i g z_i = g\sum_{i\in X} m_i z_i.$$

On the other hand, a particle of mass

$$M \coloneqq \sum_{i\in X} m_i$$

located at

$$P = \frac{1}{M}\sum_{i\in X} m_i \mathbf{r}_i$$

has potential energy

$$\widetilde{U} \coloneqq Mg\left[\frac{1}{M}\left(\sum_{i\in X} m_i\mathbf{r}_i\right)\cdot\hat{\mathbf{z}}\right] = g\sum_{i\in X} m_i(\mathbf{r}_i\cdot\hat{\mathbf{z}}) = g\sum_{i\in X} m_i z_i\,;$$

thus $U = \widetilde{U}$. ■

1.6.3.3 The Electrostatic Field

Similarly, let there be a charged body A with charge q_A at the origin. The electrostatic field due to A is the force per unit charge of a test particle, that is,

$$\mathbf{E}(\mathbf{x}) \coloneqq \frac{1}{4\pi\epsilon_0}\frac{q_A}{r^2}\hat{\mathbf{r}}$$

so that a force on a body B with charge q_B located at $\mathbf{x}$ is

$$\mathbf{F} = q_B\mathbf{E}(\mathbf{x}).$$

The electrostatic potential is

$$\phi_{\text{es}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{q_A}{r}$$

and the potential energy of B at $\mathbf{x}$ is

$$U = q_B\phi(\mathbf{x}).$$

Example N + 1

Let the two planes $x = 0$ and $x = 1$ have uniform charge density $\pm\rho$ [unit: $\mathrm{C/m^2}$], $x = 0$ being positively charged and $x = 1$ being negatively charged. It is then straightforward to show that the electric field is

$$\mathbf{E}(x) = \frac{\rho}{\epsilon_0}\hat{\mathbf{x}}$$

for all $x \in]0, 1[$. Thus, the field is *constant*, just as the local gravitational field at the surface of the Earth. The potential is therefore linear here as well:

$$\phi(x) = -\frac{\rho}{\epsilon_0}x.$$

Let there be a proton of charge $q > 0$ that is momentarily at rest at $x = 0$. It has potential energy $U = q\phi(0) = 0$ J and experiences the force $\mathbf{F} = q\mathbf{E}(0) = \frac{q\rho}{\epsilon_0}\hat{\mathbf{x}}$. When it reaches the plane $x = 1$, its potential energy has dropped to $U = q\phi(1) = -\frac{q\rho}{\epsilon_0}$, and so its kinetic energy has increased to $E_k \overset{\text{def}}{=} \frac{1}{2}mu^2 = \frac{q\rho}{\epsilon_0}$. Solving for u yields

$$u = \sqrt{2q\rho/m\epsilon_0}.$$

The final speed thus depends on both the charge and inertial mass of the proton.

1.6.3.4 The Graph of a One-Dimensional Potential

Consider a one-dimensional potential energy function $x \mapsto \phi(x)$, like the one drawn below.

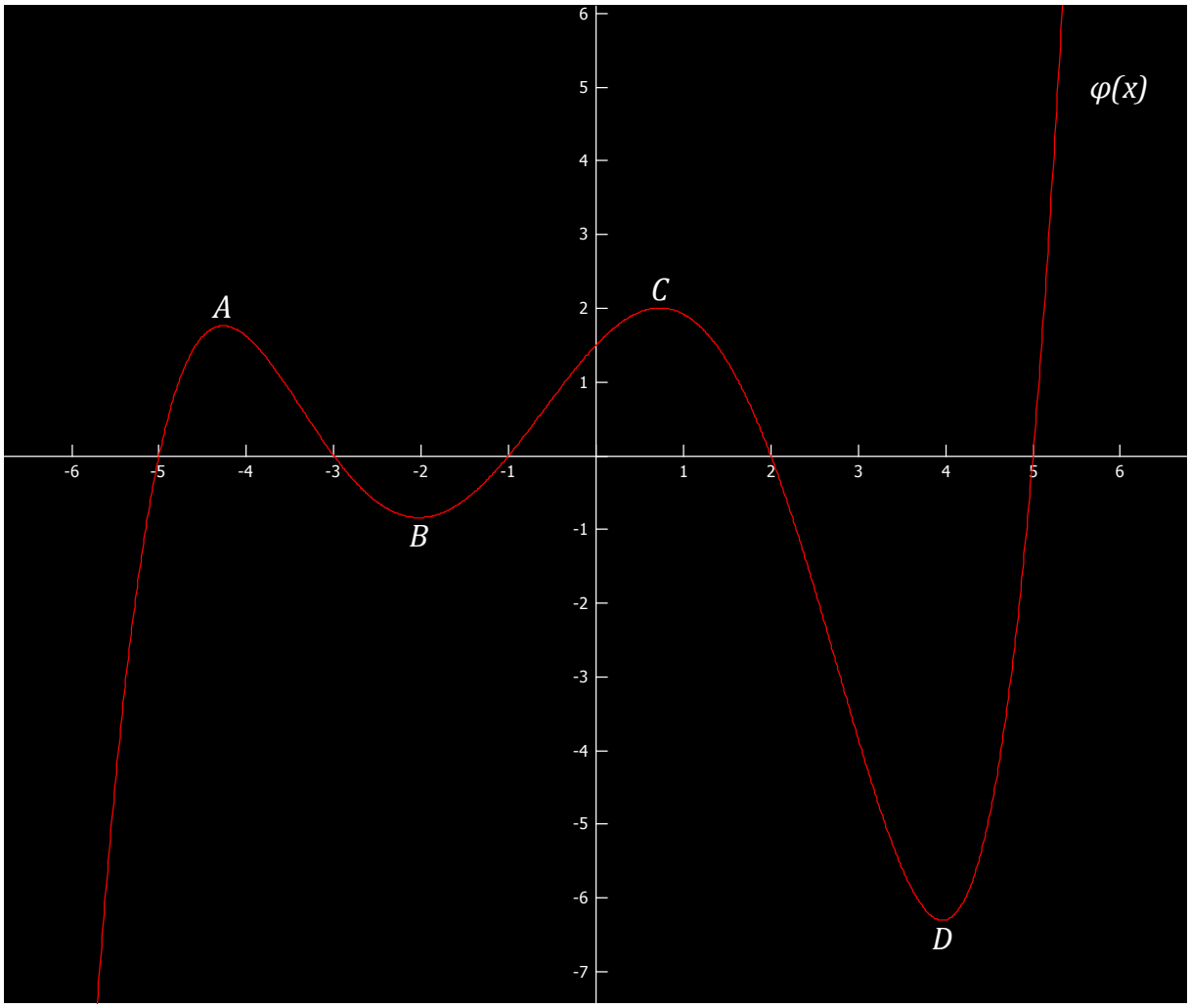


Figure 14. A one-dimensional potential energy function.

The force on a particle with this potential energy function is

$$\mathbf{F}(x) = -\nabla\phi(x) = -\frac{\partial\varphi}{\partial x}\hat{\mathbf{x}}.$$

Since this is merely a one-dimensional problem, it is rather silly to employ vectorial notation. Indeed, each vector has only one component, and so it is better to work with this scalar component alone. Thus, we write

$$F(x) = -\varphi'(x)$$

where a prime denotes a derivative with respect to x. Notice that, at a stationary point like A, B, C, and D, $\varphi'(x) = 0$ and so $F(x) = 0$. At a point with positive slope $\varphi'(x) > 0$, the force $F(x) < 0$ acts to the left, and at a point with a negative slope $\varphi'(x) < 0$, the force $F(x) > 0$ acts to the right. That is, qualitatively, the particle behaves as a ball on a hill (on the Earth) shaped like the potential, always trying to roll downwards! Local minima of the potential, such as B and D are stable equilibria, whereas local maxima, such as A and C, are unstable equilibria. Indeed, a ball at rest at B or D will remain there even if you hit it with small forces every now and then, which

does not apply to a ball at rest at A or C, in which case even a minor disturbance will make the ball move away from the instable equilibrium.

1.6.3.5 The Rollercoaster

Let's take this analogy one step further. As a model rollercoaster, let the track be two-dimensional, restricted to $z = 0$, say; that is, let the track be the image of some interval $[a, b]$ under the map $x \mapsto (x, h(x), 0)$ for some continuous 'height function' $x \mapsto h(x)$. The track might look like this:

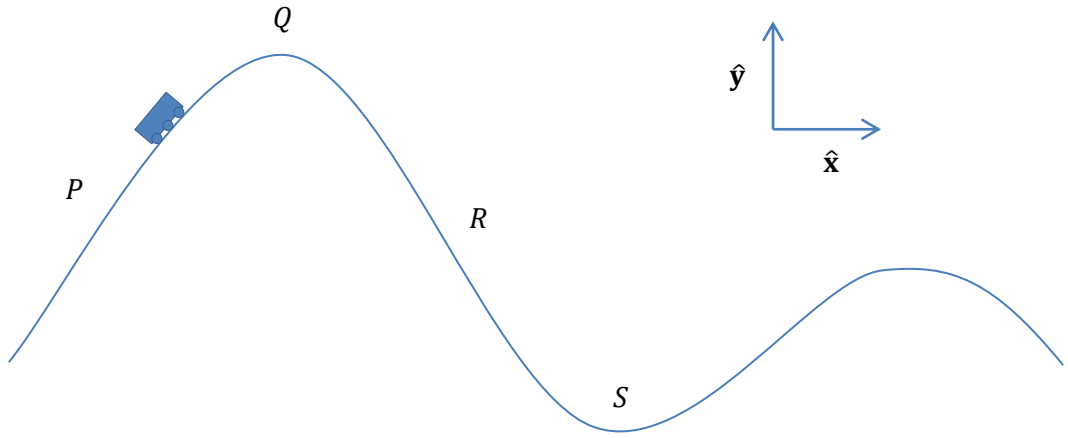


Figure 15. A rollercoaster.

Let the rollercoaster car have mass m, and assume that it performs no propulsion of its own (no motor); the only force making it move is the force of gravity. We want only to take into account the force of gravity, that is, essentially, we want to neglect the forces on the car, due to the track. Of course, we cannot just set these forces to zero, because then the car would fall right through the track, and we would not be investigating a rollercoaster at all; instead, we would be investigating a (deadly) 'drop tower' attraction. But we will still be able to *neglect* the forces. To see this, notice that, at any time, we can decompose the force $\mathbf{F}_{\text{track}}$ from the track into a part $\mathbf{F}_{\parallel}$ parallel to the track and a part $\mathbf{F}_{\perp}$ orthogonal to the track, so that $\mathbf{F}_{\text{track}} = \mathbf{F}_{\parallel} + \mathbf{F}_{\perp}$. The parallel force $\mathbf{F}_{\parallel}$ is the force of friction, which always acts in the direction opposite of the velocity, trying to slow down the car. This can be made very small, and we will simply assume it is zero. On the other hand, the force $\mathbf{F}_{\perp}$ is what is keeping the car to the track. This we cannot assume to be zero, but since it is always perpendicular to the velocity, it does no work, and so we can neglect it too when discussing the kinetic energy of the car. Consequently, the only force affecting the speed of the car is the force of gravity. The gravitational potential energy is

$$U(x) = mgh(x),$$

since the car necessarily is at height $h(x)$ at x. Now, since the only force that is doing work on the car is the force of gravity, which is a conservative force, the total mechanical energy E_t is constant. Thus the kinetic energy of the car is $E_k(x) = \frac{1}{2}mu^2 = E_t - U(x) = E_t - mgh(x)$ when the car is at x. That is, the speed u is only a function of $h(x)$. The tangential force on the car is

$$\mathbf{F}_{\text{tangent}} = -\nabla U \cdot \hat{\mathbf{t}}$$

where $\hat{\mathbf{t}}$ is the unit tangent vector. In particular, a point on the track where the tangent is horizontal, such as at Q or S, the tangential force

$$\mathbf{F}_{\text{tangent}} = -\nabla U \cdot \hat{\mathbf{t}} = -\nabla U \cdot \hat{\mathbf{x}} = -\frac{\partial U}{\partial x} = -mgh'(x) = 0.$$

That is, if the car is placed at rest at Q or S, it will remain at rest there.

1.6.3.6 The Ideal Spring

Let us return to the ideal spring. The force on the object attached to the free end of the spring is

$$\mathbf{F} = -kx\hat{\mathbf{x}}.$$

Apparently, we can forget about the spring and just think of the situation as a force field $\mathbf{F}(\mathbf{x}) = -kx\hat{\mathbf{x}}$ that affects the object. The potential energy is

$$U(x) = \frac{1}{2}kx^2.$$

Suppose that the spring is stretched so that the object is at rest at $x = \tilde{x} > 0$. Then the total mechanical energy of the object is $E_t = E_k + U(\tilde{x}) = 0 + \frac{1}{2}k\tilde{x}^2$. Then we let go of the spring. When the object flies past the origin, where the potential energy is zero, it has the maximum kinetic energy $E_k = \frac{1}{2}mu^2 = E_t - U(0) = \frac{1}{2}k\tilde{x}^2$ and maximum speed $v = \sqrt{k\tilde{x}^2/m} = \sqrt{k/m}\,\tilde{x}$. Compare with the result obtained from pure kinematics in Section 1.4.4.

The potential (↑) is drawn to the right ($k = 1$). The *main* importance of the ideal spring potential is perhaps not that engineers and scientists use ideal springs in a literal sense every day (although that is an *important* application of the potential). Rather, the ideal spring potential appears very often in physics and engineering, because it is a very 'generic' potential. Indeed, consider *any* analytic potential function φ in one dimension. Assume that it has local minimum at some point; this point is naturally a very interesting point, since it represents a stable equilibrium. Choose the x axis in such a way that this local minimum occurs at the origin. Then

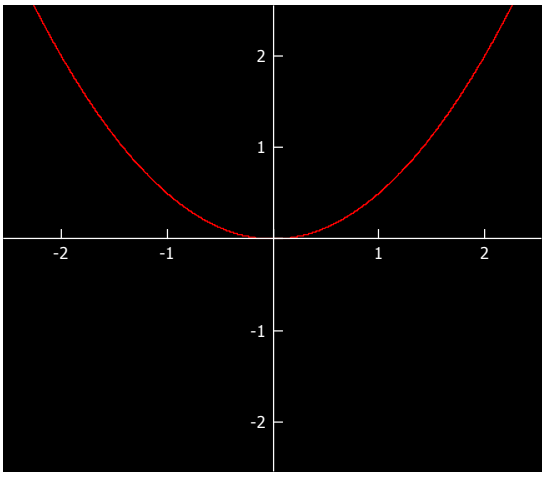


Figure 1. The spring potential.

$$\varphi(x) = \varphi(0) + \varphi'(0)x + \frac{1}{2}\varphi''(0)x^2 + \cdots.$$

Since the potential is only defined up to an additive constant, we may set $\varphi(0) = 0$. In addition, since the origin is a stationary point, $\varphi'(x) = 0$. Neglecting third-order and higher terms, we thus end up with

$$\varphi(x) \sim \frac{1}{2}kx^2$$

near the origin, where the constant $k \coloneqq \varphi''(0)$. That is, the ideal spring potential approximates essentially *any* local minimum of a generic potential! In addition, since the ideal spring potential has sines and cosines as the kinematic solutions, this also explains why sines and cosines are so abundant in physics.

1.6.3.7 The Centripetal Force

In Section 1.4.3 we found that if a particle is moving with constant speed in a circle (in some plane), then the net force on the particle has to point towards the centre of the circle, and it has to be of magnitude mv^2/r where m is the mass of the particle, v is its speed, and r is the radius of the circle. We now 'understand' that the speed of the particle is a constant of motion because the force does no work on it, since it is always orthogonal to the velocity of the particle.

1.6.4 Other Types of Energy

So far we have encountered two 'types' of energy: kinetic energy and potential energy. The latter category is made up of a large number of subcategories. Indeed, to every kind of conservative force, there is a kind of potential energy, because any such force can do work and by means of this increase the kinetic energy of a particle affected by the force, and so there is 'potential' energy, and since the field is conservative, the potential energy is a well-defined function of the spatial variable. We have already encountered gravitational potential energy, electrostatic potential energy, and the potential energy associated with an ideal spring. We also found that the 'total' energy – defined as the sum of kinetic and potential energy – was always conserved. In this section, we will discuss other forms of energy, and conclude that *the total energy is always conserved*.

- **Thermal Energy** is the kinetic energy associated with the internal microscopic random (that is, 'thermal') motion in a solid, liquid, or gaseous object. The simplest case is that of an ideal gas, that is, a gas where the gas particles do not affect each other by forces other than at perfectly elastic collisions and have no internal degrees of freedom (for instance, they do not rotate). In this case, the average kinetic energy $\frac{1}{2}mv^2$ of a gas particle is related to the temperature T of the gas according to $\frac{1}{2}mv^2 = \frac{3}{2}k_B T$ where k_B is Boltzmann's constant.[15]
- **Electromagnetic Energy** is the energy associated with an electromagnetic field. Really, there is nothing fundamentally new going on here, since the electromagnetic field is a couple of force fields, and as such, contains potential energy. The new thing is that, contrary to the simple electrostatic field and the (classical) gravitational field, this field can propagate as a wave in space, and hence energy is allowed to move from one point to another as electromagnetic radiation.

We have considered conservative forces in quite some detail. Non-conservative forces are forces that make an object *lose* total mechanical energy. Energy, however, is not lost. Instead, it appears in other forms, such as thermal or electromagnetic energy. Consider a ball released from a height $h > 0$ above the ground. In a highly idealised situation, it will bounce and come back to the initial height h *ad infinitum*; that is, kinetic energy and potential energy will be converted to and from each other – the kinetic energy being zero at the top and the potential energy being zero at the ground – while the total mechanical energy remains constant.

In a real situation, however, the ball will hit the ground, and it will lose mechanical energy. It might bounce and reach some new height $h_1 < h$, and then bounce again up to some height $h_2 < h_1$ and so on, until it finally is lying at rest on the ground. At each impact, it loses mechanical energy. Such an event probably will increase the thermal energy of the ball and the ground at the point of impact. In addition, dust particles on the ground hit by the ball might be given kinet-

[15] This result follows remarkably easily from a statistical physics approach to thermal physics.

ic energy and 'fly away'. Eventually, those particles will also come to rest, and in the final situation, essentially all mechanical energy has been converted into thermal energy. Still, the total energy contained inside the lab (if property isolated) will remain constant. This, basically, follows since every microscopic force of interaction, between every pair of atoms and/or molecules in the system, is conservative.

Another typical example is a block (with no propulsion of its own) sliding on a horizontal track. Due to friction from the surface, (again, this is a macroscopic manifestation of an intricate collaboration between a huge number of microscopic forces of electromagnetic nature), it will eventually come to a stop, if you do not push it constantly. Hence, mechanical energy is lost, and the track and block will be heated. On the other hand, if you push it with an appropriate constant force, it will eventually reach a state of constant speed, where the friction vector is the exact opposite of the force you supply.

1.6.5 Many-Particle Systems: Decomposition of Kinetic Energy

We will now investigate how the kinetic energy of a system of particles differs between different frames of reference. To this end, consider a system of N particles with 'index set' X. Let m_i be the mass of the ith particle, and let $\mathbf{u}_i$ be its velocity relative to a frame $\mathcal{F}1$. In this frame, the kinetic energy of the ith particle is

$$E_i = \frac{1}{2} m_i u_i^2$$

and so the total kinetic energy of the system is

$$E = \sum_{i \in X} E_i = \sum_{i \in X} \frac{1}{2} m_i u_i^2.$$

Let $\mathcal{F}2 \in \mathcal{F}$ be the CM frame of the system, the origin of which has coordinates $P(t)$ relative to $\mathcal{F}1$ at time t. Let $\mathbf{r}_i'$ be the position vector of the ith particle relative to $\mathcal{F}2$, so that $\mathbf{r}_i(t) = P(t) + \mathbf{r}_i'(t)$. Then the total kinetic energy of the system, as seen from $\mathcal{F}1$, is

$$\begin{aligned} E = \sum_{i \in X} \frac{1}{2} m_i u_i^2 = \sum_{i \in X} \frac{1}{2} m_i |\mathbf{u}_i|^2 = \sum_{i \in X} \frac{1}{2} m_i \left|\dot{P} + \mathbf{u}_i'\right|^2 = \sum_{i \in X} \frac{1}{2} m_i \left(\dot{P} + \mathbf{u}_i'\right)^2 = \\ = \sum_{i \in X} \frac{1}{2} m_i \left(\dot{P}^2 + 2\dot{P} \cdot \mathbf{u}_i' + \mathbf{u}_i'^2\right) = \sum_{i \in X} \frac{1}{2} m_i \dot{P}^2 + \sum_{i \in X} \frac{1}{2} m_i \left(2\dot{P} \cdot \mathbf{u}_i'\right) + \sum_{i \in X} \frac{1}{2} m_i u_i'^2. \end{aligned}$$

The middle term vanishes, for

$$\sum_{i \in X} \frac{1}{2} m_i \left(2\dot{P} \cdot \mathbf{u}_i'\right) = \dot{P} \cdot \sum_{i \in X} m_i \mathbf{u}_i' = \dot{P} \cdot \sum_{i \in X} \mathbf{p}_i' = \dot{P} \cdot \mathbf{p}' = \dot{P} \cdot \mathbf{0} = \mathbf{0}$$

because the momentum is zero as measured in the centre of mass frame. The first term, on the other hand, is simply

$$\sum_{i \in X} \frac{1}{2} m_i \dot{P}^2 = \frac{1}{2} M \dot{P}^2$$

where $M \coloneqq \sum_{i \in X} m_i$ is the total mass of the system. Hence,

$$E = \frac{1}{2} M \dot{P}^2 + \sum_{i \in X} \frac{1}{2} m_i u_i'^2.$$

Notice that $E' = \sum_{i \in X} \frac{1}{2} m_i u_i'^2$ is the kinetic energy as measured in $\mathcal{F}2$, the centre of mass frame of the system. This is the *intrinsic* kinetic energy of the system. [Notice that $E = E'$ if $\mathcal{F}1 \sim \mathcal{F}2$, that is, if $\dot{P} = \mathbf{0}$.] On the other hand, $\frac{1}{2} M \dot{P}^2$ is the kinetic energy of a particle of mass M moving with speed $\dot{P}$. This is the *extrinsic* kinetic energy of the system, that is, the kinetic energy of the system when it is considered as a point particle of mass M located at its centre of mass.

> **Exercise:** We have shown that the total kinetic energy of a system of particles, as seen from some inertial frame, can be written as the sum of the kinetic energy of the system treated as a point particle plus the 'intrinsic' kinetic energy of the system, that is, the kinetic energy as seen from the CM of the system. In the section about momentum, we also treated multi-particle systems and different frames of reference, but we gave no similar decomposition explicitly. Why?

1.7 Rotational Motion

We will review the main results regarding rotational motion.

1.7.1 Moment of Inertia, or 'Rotational Mass'

Consider a system of N particles that are all rotating about the z axis with the *same* angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ and have zero velocity in the $\hat{\mathbf{z}}$ direction. In the continuous limit $N \to \infty$, this becomes a model of a rigid body rotating about the z axis. As usual, let $X = \{1, 2, \dots, N\}$ be the set of particle indices and let $\mathbf{r}_i(t) = (x_i(t), y_i(t), z_i)$ be the position of the ith particle, which has mass m_i, at time t. Let $d_i \coloneqq \sqrt{x_i^2 + y_i^2}$ be the radial distance from the z axis to the particle. Then the speed of the particle is $v_i \coloneqq d_i\omega$ and its kinetic energy becomes

$$E_i \coloneqq \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i(d_i\omega)^2.$$

Consequently, the total kinetic energy of the system is

$$E_k \coloneqq \sum_{i\in X} E_i = \sum_{i\in X}\frac{1}{2}m_i(d_i\omega)^2 = \frac{1}{2}I\omega^2$$

where

$$I \coloneqq \sum_{i\in X} m_i d_i^2$$

is the *moment of inertia of the system [relative to the z axis]*. The continuous case is now obvious: For a rigid body occupying a volume $D \subset \mathbb{R}^3$, the moment of inertia relative to the z axis is

$$I = \iiint_D d(\mathbf{x})^2\rho(\mathbf{x})dV$$

where $\rho(\mathbf{x})$ is the density at $\mathbf{x} \in D$. The kinetic energy of this body, due to its rotation about the z axis, is

$$E_k = \frac{1}{2}I\omega^2$$

where ω is the angular speed of the body. Notice that the moment of inertia plays the same role when it comes to rotational motion as the (inertial) mass does when it comes to translational motion. And just like mass, the moment of inertia is an additive property; that is, if the rigid body consists of two disjoint parts, A and B, and these parts have moments I_A and I_B, then the moment of inertia I of the entire body is $I = I_A + I_B$. Indeed, if the volumes occupied by these two components are D_A and D_B, then $D_A \cup D_B = D$ and $D_A \cap D_B = \emptyset$ so that

$$I \overset{\text{def}}{=} \iiint_D \rho(\mathbf{x})d(\mathbf{x})^2dV = \iiint_{D_A} \rho(\mathbf{x})d(\mathbf{x})^2dV + \iiint_{D_B} \rho(\mathbf{x})d(\mathbf{x})^2dV \overset{\text{def}}{=} I_A + I_B.$$

Be sure to notice that the moment of inertia is not an intrinsic property of a rigid body. Instead, it is a property of a rigid body *and* a chosen axis of rotation.

Behold the beauty of the theory:

Linear motion:	E_k	$=$	$\frac{1}{2}m$	v^2

Rotational motion:	E_k	$=$	$\frac{1}{2}I$	ω^2
	Kinetic energy		(Half) inertia; resistance against a change of motion	Kinematic measure of motion (squared)

1.7.1.1 Examples

CYLINDER: Let us compute the moment of inertia of a homogeneous cylinder $D \subset \mathbb{R}^3$ with radius R and height h with respect to its axis of symmetry. By definition,

$$I = \iiint_D d(\mathbf{x})^2 dm = \iiint_D d(\mathbf{x})^2 \rho dV.$$

Introduce cylindrical coordinates (r, φ, z). In $r\varphi z$-space the region D corresponds to $E = [0, R] \times [0, 2\pi[\times [0, h]$ (say), $dV = r dr d\varphi dz$, and $d(r, \varphi, z) = r$ so that

$$I = \iiint_E \rho r^3 d\, r d\varphi dz = \rho \int_0^R r^3 dr \int_0^{2\pi} d\varphi \int_0^h dz = \rho \cdot \frac{1}{4} R^4 \cdot 2\pi \cdot h = \frac{\pi}{2} \rho R^4 h = \frac{1}{2} M R^2$$

where $M \coloneqq \rho \cdot R^2 \pi h$ is the total mass of the body.

BALL: A ball $D \subset \mathbb{R}^3$ of radius R centred at the origin with respect to the z axis. Introduce spherical coordinates (r, θ, φ). The set D in xyz-space corresponds to the set $E = [0, R] \times [0, \pi] \times [0, 2\pi[$ in $r\theta\varphi$-space, and $dV = r^2 \sin\theta \, dr d\theta d\varphi$. Now the radial distance

$$d(r, \theta, \varphi) = r \sin\theta$$

and so

$$I \overset{\text{def}}{=} \iiint_D d(\mathbf{x})^2 dm = \iiint_D d(\mathbf{x})^2 \rho dV = \iiint_E \rho r^4 \sin^3\theta \, dr d\theta d\varphi = \rho \int_0^R r^4 dr \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} d\varphi =$$
$$= \rho \cdot \frac{1}{5} R^5 \cdot \frac{4}{3} \cdot 2\pi = \frac{8\pi}{15} \rho R^5 = \frac{2}{5} M R^2$$

where $M \coloneqq \rho \cdot \frac{4}{3} \pi R^3$ is the mass of the ball.

1.7.2 Angular Momentum

Let $\mathcal{F}1 \in \mathcal{F}$ be a frame of reference, and let A be a particle with mass m, position $\mathbf{r}$ and velocity $\mathbf{v}$. The quantity

$$\mathbf{J} \coloneqq \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

is called the *angular momentum* of the particle, where $\mathbf{p}$ is the (linear) momentum of the particle. For a system of particles, the angular momentum is defined as the sum of the angular momenta of the individual particles. In our usual notation, we write

$$\mathbf{J} \coloneqq \sum_{i \in X} \mathbf{J}_i$$

where $\mathbf{J}$ is the total angular momentum of a system of N particles with 'index set' X, in which the angular momentum of the ith particle is $\mathbf{J}_i$. The analogous definition in the continuous case is obvious.

Now we restrict our attention to a very important special case, namely, the case of a rigid body rotating about one of its axes of symmetry. To make this precise, we define

Definition

Let A_θ be the linear map

$$A_\theta: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

that is, a rotation θ radians about the z axis (in the positive sense). If $X \subset \mathbb{R}^3$ then we denote the image of X under a linear map A by $A(X)$, that is,

$$A(X) \coloneqq \{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y} = A(\mathbf{x}), \mathbf{x} \in X\}.$$

Consider a rigid body A occupying a volume $D \subset \mathbb{R}^3$. Let $\rho(\mathbf{x})$ be the density of A at $\mathbf{x} \in D$. The rigid body A is *symmetric about the z axis* iff

$$A_\theta(D) = D, \qquad \forall \theta \in \mathbb{R}$$

and

$$\rho\big(A_\theta(\mathbf{x})\big) = \rho(\mathbf{x}), \qquad \forall (\theta, \mathbf{x}) \in \mathbb{R} \times D.$$

Consider a rigid body A that is symmetric about the z axis. The total angular momentum is

$$\mathbf{J} \overset{\text{def}}{=} \iiint_D \big(\mathbf{r}(\mathbf{x}) \times \mathbf{v}(\mathbf{x})\big)\rho(\mathbf{x})dV.$$

A general point $\mathbf{x}$ on the body is shown below, together with the direction of the cross product $\mathbf{r} \times \mathbf{v}$, where the velocity vector $\mathbf{v}$ is pointing into the page.

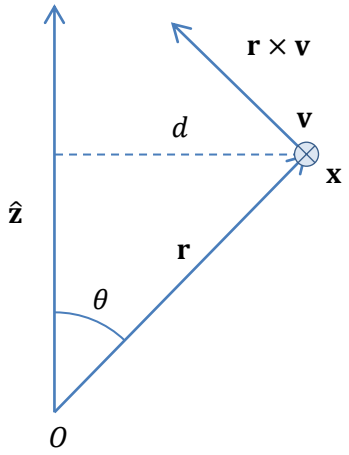


Figure 16. The cross product r × v.

Since the body is symmetric about the z axis, the projection onto the xy-plane of the integral (which is a vector) must vanish. Hence, it will suffice to compute only the z component of the integral, which is given by the integral of the projection of the integrand to the z axis; in symbols,

$$\mathbf{J} = \left(\iiint_D ([\mathbf{r}(\mathbf{x}) \times \mathbf{v}(\mathbf{x})] \cdot \hat{\mathbf{z}})\rho(\mathbf{x})dV\right)\hat{\mathbf{z}}.$$

Now

$$|\mathbf{r}(\mathbf{x}) \times \mathbf{v}(\mathbf{x})| = |\mathbf{r}(\mathbf{x})||\mathbf{v}(\mathbf{x})|$$

since $\mathbf{r}(\mathbf{x}) \perp \mathbf{v}(\mathbf{x})$. Furthermore,

$$v \coloneqq |\mathbf{v}(\mathbf{x})| = \omega d$$

where ω is the angular speed, and the radial distance between the z axis and $\mathbf{x}$ is

$$d = r \sin\theta.$$

Consequently,

$$|\mathbf{r}(\mathbf{x}) \times \mathbf{v}(\mathbf{x})| = |\mathbf{r}(\mathbf{x})||\mathbf{v}(\mathbf{x})| = r\omega d = r^2 \omega \sin\theta,$$

whence[16],

$$[\mathbf{r}(\mathbf{x}) \times \mathbf{v}(\mathbf{x})] \cdot \hat{\mathbf{z}} = r^2 \omega \sin\theta \cdot \sin\theta = r^2 \omega \sin^2\theta.$$

Thus,

$$\mathbf{J} = \left(\iiint_D r^2 \omega \sin^2\theta \, \rho(\mathbf{x}) dV \right) \hat{\mathbf{z}} = \left(\iiint_D r^2 \sin^2\theta \, \rho(\mathbf{x}) dV \right) \omega \hat{\mathbf{z}} = \left(\iiint_D d(\mathbf{x})^2 \rho(\mathbf{x}) dV \right) \boldsymbol{\omega} = I\boldsymbol{\omega}.$$

That is, we have found

Proposition

Let A be a rigid body that is *symmetric and rotating* about the z axis, and denote its moment of inertia and angular velocity by I and $\boldsymbol{\omega}$, respectively. Then the angular momentum of A is

$$\mathbf{J} = I\boldsymbol{\omega}.$$

Behold the beauty of the theory:

Linear motion:	$\mathbf{p}$	$=$	m	$\mathbf{v}$
Rotational motion:	$\mathbf{J}$	$=$	I	$\boldsymbol{\omega}$
(symmetric case only)	Dynamic measure of motion		Inertia; resistance against a change of motion	Kinematic measure of motion

However, this time the symmetry is not perfect. Indeed, the equation $\mathbf{J} = I\boldsymbol{\omega}$ only holds for a rigid body that is symmetric about the axis about which it is rotating.

1.7.3 Torque

Let A be a particle with mass m located at $\mathbf{r}$, and let $\mathbf{F}$ be the net force on the particle. The quantity

$$\boldsymbol{\tau} \coloneqq \mathbf{r} \times \mathbf{F}$$

is called the *torque* on the particle. The torque on a system of particles is defined as the sum of the torques of the individual particles, that is,

$$\boldsymbol{\tau} \coloneqq \sum_{i \in X} \boldsymbol{\tau}_i.$$

[16] Or $r^2 \omega \sin^2\theta \cdot 1 \cdot \cos\left(\frac{\pi}{2} - \theta\right)$ if you prefer to think of the scalar product that way.

We have

Proposition

Consider a system of N particles, and denote the angular momentum and the torque on the system by $\mathbf{J}$ and $\boldsymbol{\tau}$, respectively. Then

$$\boldsymbol{\tau} = \frac{d\mathbf{J}}{dt}.$$

which could also serve as the definition of 'torque'.

Proof

$$\frac{d\mathbf{J}}{dt} = \frac{d}{dt}\left(\sum_{i\in X} \mathbf{r}_i \times m\mathbf{v}_i\right) = \sum_{i\in X} \frac{d}{dt}(\mathbf{r}_i \times m\mathbf{v}_i) = \sum_{i\in X} (\mathbf{v}_i \times m\mathbf{v}_i + \mathbf{r}_i \times m\mathbf{a}_i) = \sum_{i\in X} \mathbf{r}_i \times \mathbf{F}_i \overset{\text{def}}{=} \boldsymbol{\tau}.$$

■

Just as force is the rate of change of (linear) momentum, torque is the rate of change of angular momentum. If we are dealing with a rigid body symmetric and rotating about the z axis, then we also have

$$\boldsymbol{\tau} = \frac{d\mathbf{J}}{dt} = I\boldsymbol{\alpha}$$

where $\boldsymbol{\alpha} \coloneqq d\boldsymbol{\omega}/dt$ is the angular acceleration. That is, we have a rotational analogue of Newton's second law!

Linear motion:	$\mathbf{F}$	$=$	m	$\mathbf{a}$
Rotational motion:	$\boldsymbol{\tau}$	$=$	I	$\boldsymbol{\alpha}$
(symmetric case only)	'Force'		Inertia	Acceleration

1.7.4 Conservation of Angular Momentum

We have seen that, in any isolated system, the total (linear) momentum is a constant of motion. We will now see that the same thing applies to the total angular momentum of such a system, but in order to prove this, we will find it necessary to use a slightly stronger version of Newton's third law.

Consider an isolated system of N particles in which the force on the ith particle due to the jth particle, is $\mathbf{F}_{ij}$, and define $\mathbf{F}_{ii} = \mathbf{0}$ as usual. Since the system is isolated, there are no other forces affecting the particle. Thus the total force on the ith particle is

$$\mathbf{F}_i = \sum_{j\in X} \mathbf{F}_{ij}$$

and so the torque on this particle is

$$\boldsymbol{\tau}_i = \mathbf{r}_i \times \mathbf{F}_i = \mathbf{r}_i \times \sum_{j\in X} \mathbf{F}_{ij}.$$

Therefore, the rate of change of the total angular momentum is

$$\frac{d\mathbf{J}}{dt} = \sum_{i\in X} \boldsymbol{\tau}_i = \sum_{i\in X}\left(\mathbf{r}_i \times \sum_{j\in X} \mathbf{F}_{ij}\right) = \sum_{(i,j)\in X^2} \mathbf{r}_i \times \mathbf{F}_{ij} = \sum_{i\in X}\left(\sum_{i<j}\left(\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}\right)\right) =$$
$$= \sum_{i\in X}\left(\sum_{i<j}\left(\mathbf{r}_i \times \mathbf{F}_{ij} - \mathbf{r}_j \times \mathbf{F}_{ij}\right)\right) = \sum_{i\in X}\left(\sum_{i<j}\left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{F}_{ij}\right).$$

Now, we have to introduce

Hypothesis (The Strong Newton's Third Law)

Let A and B be two particles located at $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively. If A is affecting B with force $\mathbf{F}_{AB}$, then B is affecting A with the force $\mathbf{F}_{BA} = -\mathbf{F}_{AB}$ and $\mathbf{F}_{AB}$ (and $\mathbf{F}_{BA}$) is parallel with the displacement $\mathbf{r}_B - \mathbf{r}_A$.

In fact, when one imagines Newton's third law (for instance, using a picture as in Figure 7), one does usually assume the parallelism between the force vectors and the displacement vector. In addition, most forces we know of satisfy the strong version of the law (Newton's law of universal gravitation, Coulomb's law, etc.). Using this strong form, it is obvious that

$$\left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{F}_{ij} = \mathbf{0}, \qquad \forall (i,j) \in X^2.$$

Thus

$$\frac{d\mathbf{J}}{dt} = \mathbf{0}$$

and we have shown

Theorem

Assuming the strong form of Newton's third law, the total angular momentum of an isolated system of discrete particles is constant in time.

1.7.5 Decomposition of Angular Momentum

Consider any system of particles, and let $\mathcal{F}1$ be a frame relative to which the centre of mass is located at $P(t)$ at time t. Let $\mathbf{r}_i$ be the position of the ith particle, with mass m_i, relative to $\mathcal{F}1$, and let $\mathbf{r}_i'$ be the position relative to the centre of mass. Then

$$\mathbf{r}_i(t) = P(t) + \mathbf{r}_i'(t)$$

where

$$P(t) = \frac{1}{M}\sum_{i\in X} m_i \mathbf{r}_i.$$

Relative to $\mathcal{F}1$ the total angular momentum is

$$\mathbf{J} \overset{\text{def}}{=} \sum_{i\in X} \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_{i\in X}\left(P(t) + \mathbf{r}_i'(t)\right) \times m_i\left(\dot{P}(t) + \mathbf{v}_i'(t)\right) =$$
$$= \sum_{i\in X} P(t) \times m_i \dot{P}(t) + \sum_{i\in X} P(t) \times m_i \mathbf{v}_i'(t) + \sum_{i\in X} \mathbf{r}_i'(t) \times m_i \dot{P}(t) +$$
$$+ \sum_{i\in X} \mathbf{r}_i'(t) \times m_i \mathbf{v}_i'(t).$$

The two middle terms vanish. Indeed,

$$\sum_{i\in X} P(t)\times m_i\mathbf{v}_i'(t) = P(t)\times\sum_{i\in X} m_i\mathbf{v}_i'(t) = P(t)\times\mathbf{p}'(t) = P(t)\times\mathbf{0} = \mathbf{0}$$

since the total momentum $\mathbf{p}'$ is zero as seen from the centre of mass frame (Proposition NN), and

$$\sum_{i\in X}\mathbf{r}_i'(t)\times m_i\dot{P}(t) = \left(\sum_{i\in X} m_i\mathbf{r}_i'(t)\right)\times\dot{P}(t) = \mathbf{0}\times\dot{P}(t) = \mathbf{0}$$

because $\left(\frac{1}{M}\right)\sum_{i\in X} m_i\mathbf{r}_i'(t)$ are the coordinates of the centre of mass expressed in the CM frame. Thus,

$$\mathbf{J} = \sum_{i\in X} P(t)\times m_i\dot{P}(t) + \sum_{i\in X}\mathbf{r}_i'(t)\times m_i\mathbf{v}_i'(t) = P\times M\dot{P} + \sum_{i\in X}\mathbf{r}_i'(t)\times m_i\mathbf{v}_i'(t),$$

that is, the total angular momentum is decomposed into two parts: the total angular momentum $P\times M\dot{P}$ of the system as a whole [considered as a point particle of mass M at its centre of mass P] and the total angular momentum $\sum_{i\in X}\mathbf{r}_i'(t)\times m_i\mathbf{v}_i'(t)$ due to the internal structure of the system. If $\mathcal{F}1\sim\mathcal{F}2$, where $\mathcal{F}2$ is the CM frame, then $\dot{P} = \mathbf{0}$ and

$$\mathbf{J} = \sum_{i\in X}\mathbf{r}_i'(t)\times m_i\mathbf{v}_i'(t) \overset{\text{def}}{=} \mathbf{J}',$$

that is, the quantity $\sum_{i\in X}\mathbf{r}_i'(t)\times m_i\mathbf{v}_i'(t)$ is really the *intrinsic* angular momentum of the system in the sense that it is the angular momentum one would measure in the rest frame of the centre of mass of the system. In any other frame, one obtains the total angular momentum by adding the *extrinsic* angular momentum $P\times M\dot{P}$ to the intrinsic angular momentum, which is due to the motion of the centre of mass, that is, to the motion of the system considered as a particle without internal structure.

For instance, consider a planet orbiting a star. The total angular momentum of the planet is then

$$\mathbf{J} = \mathbf{J}_{\text{orbit}} + \mathbf{J}_{\text{spin}}$$

where $\mathbf{J}_{\text{orbit}} \coloneqq P\times M\dot{P}$ is the angular momentum due to the orbit about the star, and $\mathbf{J}_{\text{spin}} \coloneqq \sum_{i\in X}\mathbf{r}_i'(t)\times m_i\mathbf{v}_i'(t)$ is the angular momentum due to the planet's spin about its own axis.

Notice that, in the special case of a rigid body spinning about a symmetry axis,

$$\mathbf{J} = P\times M\dot{P} + I\boldsymbol{\omega}$$

where I is the moment of inertia (about the symmetry axis) and $\boldsymbol{\omega}$ is the angular velocity of the spin.

1.7.6 Decomposition of the Kinetic Energy

We will use the same setup as in the last section, but instead of analysing the angular momentum, we will investigate the kinetic energy. To this end, we can employ the results obtained in Section 1.6.5. Hence, let $\mathcal{F}2\in\mathcal{F}$ be the centre of mass frame of the system, in which the kinetic energy is E_{internal}. This is the energy associated with the motion inside the system. In particular, this contains any kinetic energy caused by spin. Let $\mathcal{F}1\in\mathcal{F}$ be any inertial frame, relative to which $\mathcal{F}2$ is moving with speed v. If M is the total mass of the system, then

$$E_k = \frac{1}{2}Mv^2 + E_{\text{internal}}$$

is the total kinetic energy of the system as seen from $\mathcal{F}1$. In particular, if the system is a rigid body rotating about a symmetry axis, then

$$E_k = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

where I is the moment of inertia and ω is the angular speed. For example, the body could be a planet orbiting the sun. Then $\frac{1}{2}Mv^2$ is the kinetic energy of the planet – considered as a point particle – due to its motion about the sun, whereas $\frac{1}{2}I\omega^2$ is the 'internal' kinetic energy of the planet, due to its rotation about its own axis.

1.7.7 Example: A Rolling Stone

Consider a ball-shaped stone of radius r and mass m that is rolling (without slipping) down a hill. Of course, we will not ignore friction, because this is what makes the stone roll. However, due to the rolling, the lost mechanic energy is very small. In fact, we will assume that no mechanic energy is lost at all. Let the hill be of height h and inclination α, as indicated in Figure 17.

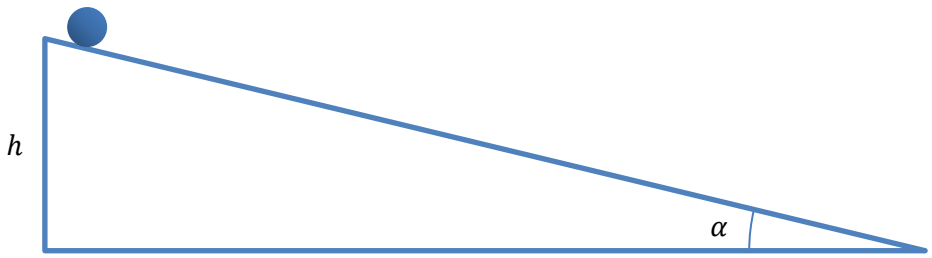


Figure 17. A ball rolling down an inclined hill.

According to Proposition NN, the gravitational potential energy of the ball is the same as the gravitational potential energy of a point particle of the same mass located at the ball's centre of mass. This potential energy is simply mgy at height y. The difference in potential energy between the top and the bottom of the hill is thus $U \coloneqq mgh$; since we assume that the ball is at rest at the top of the hill, this is the total mechanical energy of the system.

At the bottom of the hill, the total mechanical energy mgh is divided into translational and rotational kinetic energy in an additive fashion according to the last section. That is,

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

where v is the (final) translational speed of the ball, ω is its (final) rotational angular speed, and the moment of inertia

$$I = \frac{2}{5}mr^2$$

according to the Examples above. At first sight, it might look like v and ω are two independent variables, but they are not. Indeed, given a fixed speed v, the ball has to rotate with a certain angular speed ω. To find the quantitative relation, assume that a ball of radius r is rolling along a straight line and has travelled a distance $x(t)$ at time t. By then it must have made $x(t)/2\pi r$ full revolutions [since $x(0) = 0$], or a rotation of

$$\theta(t) = 2\pi \cdot x(t)/2\pi r = x(t)/r$$

radians about its axis. Differentiation yields

$$\omega(t) \overset{\text{def}}{=} \dot{\theta}(t) = \dot{x}(t)/r =: v(t)/r.$$

Substitution in (↑) yields

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{r^2}$$

which is solved with respect to v to obtain

$$v = \sqrt{\frac{2mghr^2}{mr^2 + I}} = \sqrt{\frac{10}{7}gh}.$$

Had we not taken the rotation into account, we would have used $mgh = \frac{1}{2}mv^2$ and found $v = \sqrt{2gh}$. Thus, we can 'disregard' the effects of the rotation by letting $I \to 0$.

Notice in particular that the final speed v does not depend on either the mass or radius of the ball, not even when rotational motion is taken into account. As a concrete example, let $h = 3$ m. Then $v = \sqrt{10gh/7} = 6.48\,\text{m/s}$ when rotation is taken into account, and $v = \sqrt{2gh} = 7.67\,\text{m/s}$ when rotation is neglected. The effect is not negligible at all.

Let us end this section by examining the 'shape' of the motion. In the case of no rotational energy ($I = 0$), according to (↑) the speed is $v = \sqrt{2gh}$ when the ball has travelled a vertical distance h, the height of the plane. However, there is nothing special with this particular height. Instead, in general, the speed is $v = \sqrt{2gy}$ when the ball has travelled a vertical distance y for all $y \geq 0$. A vertical distance y corresponds to a distance $x = y\csc\alpha$ along the slope, if we introduce x as a coordinate along the slope with $x = 0$ at the top. In addition, since the ball is moving along the slope, $v = dx/dt$. Hence,

$$\frac{dx}{dt} = \sqrt{2g\sin\alpha \cdot x}$$

or

$$\frac{1}{\sqrt{x}}\frac{dx}{dt} = \sqrt{2g\sin\alpha}$$

which is a separable first-order ODE, which is integrated to yield

$$2\sqrt{x} = \sqrt{2g\sin\alpha}\,t$$

when we impose the natural coordinate restriction $x(0) = 0$. Thus[17],

[17] This is an overly involved way of obtaining this simple result. Indeed, the force of gravity on the ball is $F_g = mg$ and its component along the track is $F = mg\sin\alpha$. Thus the acceleration is constant,

$$x = \frac{1}{2} g \sin\alpha \ t^2,$$

that is, $x \propto t^2$ just as in the case of free fall. When the rotational energy is taken into account,

$$\frac{dx}{dt} = \sqrt{\frac{10}{7} g \sin\alpha \cdot x}$$

or

$$\frac{1}{\sqrt{x}} \frac{dx}{dt} = \sqrt{\frac{10}{7} g \sin\alpha}$$

which is integrated to yield

$$x = \frac{5}{14} g \sin\alpha \ t^2.$$

We remark that the 'shape' of the motion is the same (still $x \propto t^2$). The constant $\frac{1}{2} = 0.5$ has been replaced by $\frac{5}{14} = 0.35$ …, however, and so the speed is slightly lower at each point.

$a = g \sin\alpha \Rightarrow v = g \sin\alpha \cdot t \Rightarrow x = \frac{1}{2} g \sin\alpha \cdot t^2$ imposing $v(0) = 0$ and $x(0) = 0$. However, the 'overly involved' method works equally well in the case where the rotational energy *is* taken into account, and in this case it isn't *overly* involved. Indeed, if we also consider the rotational motion, we need to consider the force of friction, causing the torque that makes the ball rotate. Thus the *net* force will be different from $mg \sin\alpha$; it will be $mg \sin\alpha - F_{\text{friction}} < mg \sin\alpha$.

1.8 Kepler's Laws of Planetary Motion

Kepler's laws of planetary motion are three fundamental theorems concerning the motion of the planets around the Sun, discovered by Johannes Kepler (1571-1630) in the early seventeenth century. They are

1. The orbit of every planet is an ellipse with the Sun at one of its foci.
2. The radius vector from the sun to a planet sweeps out equal areas during equal intervals of time.
3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its elliptic orbit. The constant of proportionality is a property only of the Sun.

In this section we will derive Kepler's laws from Newton's laws of motion, the latter being published almost a century after Kepler discovered his laws by investigating data obtained by the Danish astronomer Tycho Brahe (1546-1601).

1.8.1 Ellipses

Since we will be working a lot with ellipses, we will review some basic properties of these curves. (See Figure 18a.)

Definition

An *ellipse* E is a set of points (x, y) satisfying the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \qquad 0 < b \leq a$$

in a suitably positioned and oriented Cartesian coordinate system. The x and y axes are called the *major* and *minor* axes of the ellipse, and the constants a and b are called the semi-major and semi-minor axis lengths, respectively.

Notice that 'semi' in this case means 'half'. The ellipse is also the image E of the interval $I = [0, 2\pi[$ under the parameterisation map

$$\varphi \mapsto (x, y) = (a\cos\varphi\,, b\sin\varphi), \qquad \varphi \in I.$$

Clearly the ellipse is a closed curve symmetric about both its major and minor axis, and if $a = b =: r$ we obtain a circle of radius r.

Definition

Let E be an ellipse with semi-major and semi-minor axes a and b, respectively. The quantity

$$\epsilon := \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

is called the *eccentricity* of the ellipse.

Notice that the eccentricity $\epsilon \in [0, 1[$ is a dimensionless measure of the amount by which a circle has to be 'flattened' [by a linear transformation] in order to become the ellipse. Indeed, in a circle $a = b$ and so $\epsilon = 0$. At the other extreme, if $a \gg b$ then $b/a \approx 0$ and so $\epsilon \approx 1$.

Definition

The two points

$$F_1 \coloneqq -\sqrt{a^2-b^2}\hat{\mathbf{x}}, \qquad F_2 \coloneqq +\sqrt{a^2-b^2}\hat{\mathbf{x}}$$

are called the *foci* of the ellipse.

Notice that F_1 and F_2 both lie on the major axis, that $F_1 = -F_2$, and that $|F_1| = |F_2| = \sqrt{a^2-b^2} = a\sqrt{1-\left(\frac{b}{a}\right)^2} = a\epsilon < a$, displaying the fact $F_1, F_2 \in \text{int}(E)$. Now we will prove the perhaps most well-known fact about the ellipse:

Lemma

Let F_1 and F_2 be the foci of an ellipse E. Then $|P-F_1| + |F_2-P| = 2a, \forall P \in E$.

Proof

Let $P \in E$. Then $P = \left(x, \pm b\sqrt{1-\frac{x^2}{a^2}}\right)$ for some $x \in [-a, a]$ and we have

$$\begin{aligned}
&|P-F_1| + |F_2-P| = \\
&= \sqrt{\left(x+\sqrt{a^2-b^2}\right)^2 + \left(\pm b\sqrt{1-\frac{x^2}{a^2}} - 0\right)^2} + \\
&+ \sqrt{\left(\sqrt{a^2-b^2}-x\right)^2 + \left(0 \mp b\sqrt{1-\frac{x^2}{a^2}}\right)^2} = \\
&= \sqrt{x^2 + 2x\sqrt{a^2-b^2} + a^2 - \frac{b^2x^2}{a^2}} + \sqrt{a^2 - 2x\sqrt{a^2-b^2} + x^2 - \frac{b^2x^2}{a^2}} = \\
&= a\left(\sqrt{\frac{x^2}{a^2} + 2\frac{x}{a}\sqrt{1-\frac{b^2}{a^2}} + 1 - \frac{b^2x^2}{a^2a^2}} + \sqrt{1 - 2\frac{x}{a}\sqrt{1-\frac{b^2}{a^2}} + \frac{x^2}{a^2} - \frac{b^2x^2}{a^2a^2}}\right) = \\
&= a\left(\sqrt{\frac{x^2}{a^2} + 2\frac{x}{a}\epsilon + 1 - (1-\epsilon^2)\frac{x^2}{a^2}} + \sqrt{1 - 2\frac{x}{a}\epsilon + \frac{x^2}{a^2} - (1-\epsilon^2)\frac{x^2}{a^2}}\right) = \\
&= a\left(\sqrt{2\frac{x}{a}\epsilon + 1 + \epsilon^2\frac{x^2}{a^2}} + \sqrt{1 - 2\frac{x}{a}\epsilon + \epsilon^2\frac{x^2}{a^2}}\right) = \\
&= a\left(\sqrt{\left(1+\frac{\epsilon x}{a}\right)^2} + \sqrt{\left(1-\frac{\epsilon x}{a}\right)^2}\right) = a\left(\left|1+\frac{\epsilon x}{a}\right| + \left|1-\frac{\epsilon x}{a}\right|\right) = \\
&= a\left(1+\frac{\epsilon x}{a} + 1 - \frac{\epsilon x}{a}\right) = 2a
\end{aligned}$$

since $|x| \leq a$ and $\epsilon \in [0, 1[$. ■

An alternative characteristic of the eccentricity is given by

Lemma

Let f be the distance from the origin to any of the two foci of an ellipse E, and let a be the semi-major axis. Then the eccentricity $\epsilon = f/a$.

Proof (again)

$$\epsilon \overset{\text{def}}{=} \sqrt{1-\left(\frac{b}{a}\right)^2} = \sqrt{\frac{a^2-b^2}{a^2}} = \frac{\sqrt{a^2-b^2}}{a} = \frac{f}{a}$$

by the definition of the foci. ■

Definition

The distance between any of the foci and the ellipse as measured along a line parallel to the minor axis is called the *semi-latus rectum* of the ellipse.

Lemma

The semi-latus rectum of an ellipse of eccentricity ϵ and semi-major axis length a is $a(1-\epsilon^2)$.

Proof

The foci are located at $x = \pm\sqrt{a^2-b^2}$ and $y = 0$. If $(x,y) \in E$ then $(x/a)^2 + (y/b)^2 = 1$ and so

$$y = \pm b\sqrt{1-\frac{x^2}{a^2}} = \pm b\sqrt{1-\frac{a^2-b^2}{a^2}} = \pm b\sqrt{\frac{b^2}{a^2}} = \frac{b^2}{a} = \pm\frac{a^2(1-\epsilon^2)}{a} = \pm a(1-\epsilon^2).$$ ■

Figure 18a shows an ellipse with $(a,b) = (2.3, 1.7)$ together with its foci. The eccentricity of this ellipse is $\epsilon = 0.67$, its focal length is 1.5, and its semi-latus rectum is 1.3.

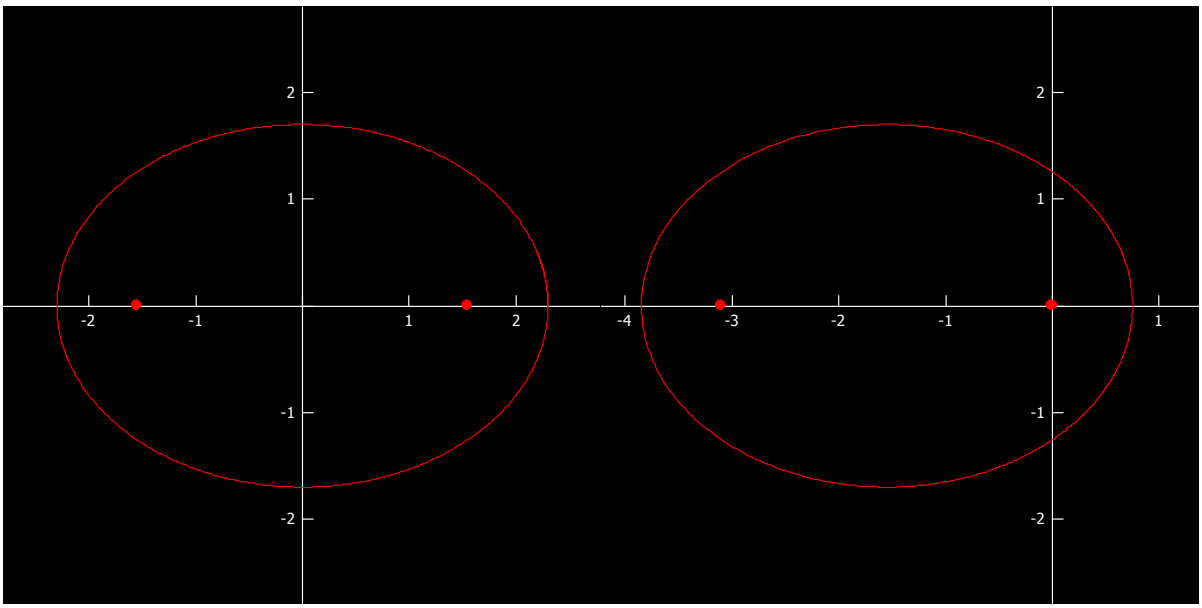


Figure 18. a) An ellipse and its foci. b) The same ellipse translated to the left so that its right focus is at the origin.

1.8.1.1 The Polar Equation

We end this subsection by giving an alternative characterisation of the ellipse, namely, the *polar* equation $r = r(\varphi)$ of an ellipse oriented the usual way but *with its right focus at the origin*. We start with the usual equation of the ellipse:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \qquad 0 < b \leq a.$$

Since the right focus is located at $F_2 = \sqrt{a^2 - b^2}\hat{\mathbf{x}}$ it follows that

$$\left(\frac{x + \sqrt{a^2 - b^2}}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

is the Cartesian equation for the same ellipse but translated so that the right focus is found at the origin instead (see Figure 18b). We now introduce polar coordinates

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

and obtain

$$\left(\frac{r\cos\varphi + \sqrt{a^2 - b^2}}{a}\right)^2 + \left(\frac{r\sin\varphi}{b}\right)^2 = 1.$$

Observe that

$$\frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}} \stackrel{\text{def}}{=} \epsilon, \qquad b = a\sqrt{1 - \epsilon^2}$$

so that the equation can be written

$$\left(\frac{r\cos\varphi}{a} + \epsilon\right)^2 + \left(\frac{r\sin\varphi}{a\sqrt{1 - \epsilon^2}}\right)^2 = 1.$$

We solve for r and find

Lemma

The polar equation of an ellipse with semi-major axis along the $\varphi = 0$ axis and the right-most focus at the origin is

$$r = \frac{a(1 - \epsilon^2)}{1 + \epsilon\cos\varphi}$$

where a is the semi-major axis length and ϵ is the eccentricity.

Again, this curve is shown in Figure 18b. We remark that the numerator in the polar equation is the semi-latus rectum of the ellipse.

1.8.2 Hyperbolae

We will also need some familiarity with *hyperbolae*, so we give a very brief treatment of these curves in this subsection. See Figure 19a.

Definition

A *hyperbola* H is a set of points (x, y) satisfying the equation

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

for some $a, b > 0$ in a suitably positioned and oriented Cartesian coordinate system. The two connected parts $H_\pm \coloneqq \{(x, y) \in H : \operatorname{sgn} x = \pm 1\}$ are called the *right* (+) and *left* (−) *branches* of H.

Notice that the branch $H_\pm$ of the hyperbola is the image of $\mathbb{R}$ under the parameterisation map

$$t \mapsto (x, y) = (\pm a \cosh t\,, b \sinh t), \qquad t \in \mathbb{R}.$$

Hence, in a sense, *the hyperbolic functions are to hyperbolae as are the trigonometric functions to ellipses*. Notice also that hyperbolae are open curves and that, far from the origin, the branches of a hyperbola are closely approximated by the straight-line asymptotes

$$y = \pm \frac{b}{a} x.$$

Definition

The quantity

$$\epsilon \coloneqq \sqrt{1 + \left(\frac{b}{a}\right)^2}$$

is called the *eccentricity* of the hyperbola.

Clearly $\epsilon > 1$, in contrast to the ellipse, where the quantity with the same name has $\epsilon < 1$.

Definition

The two points

$$F_1 \coloneqq -\sqrt{a^2 + b^2}\hat{\mathbf{x}}, \qquad F_2 \coloneqq +\sqrt{a^2 + b^2}\hat{\mathbf{x}}$$

are called the *foci* of the hyperbola.

Notice that $|F_1| = |F_2| > a$, and so the foci lie ‘inside’ each branch. Figure 19a displays a hyperbola and its foci.

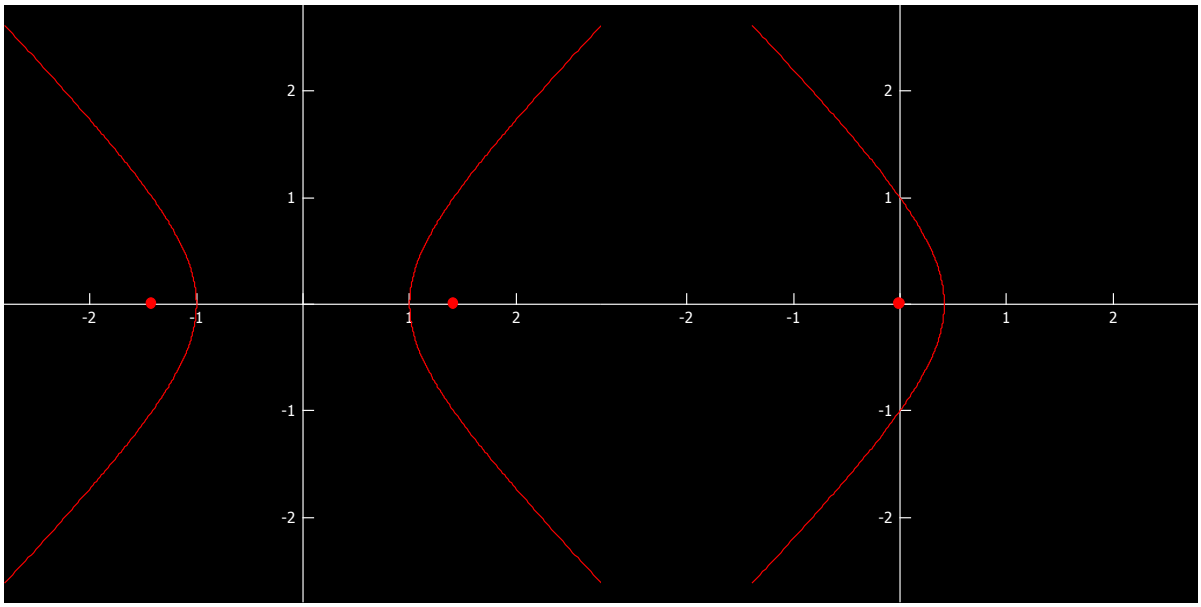


Figure 19. a) A hyperbola and its foci. b) The left branch of the same hyperbola translated to the right so that the left focus of the hyperbola is at the origin.

1.8.2.1 The Polar Equation

We will need the polar equation of the left branch H_- of the hyperbola. To this end, we start with the Cartesian equation for the left branch of the hyperbola,

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1, \qquad x < 0$$

and, just as we did with the ellipse, we translate the curve horizontally,

$$\left(\frac{x - \sqrt{a^2 + b^2}}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1, \qquad x < \sqrt{a^2 + b^2}$$

to obtain a left branch of a hyperbola with its left focus at the origin; see Figure 19b. Finally, we introduce polar coordinates,

$$\left(\frac{r\cos\varphi - \sqrt{a^2 + b^2}}{a}\right)^2 - \left(\frac{r\sin\varphi}{b}\right)^2 = 1, \qquad r\cos\varphi < \sqrt{a^2 + b^2}.$$

Solving for r yields

Lemma

The polar equation of the left branch H_- of a hyperbola H with its left focus at the origin is

$$r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon\cos\varphi}$$

where a is the semi-major axis length and ϵ is the eccentricity, and where $\epsilon\cos\varphi > -1$.

Again, this curve is shown in Figure 19b.

1.8.3 Polar Coordinates

Before continuing, we will derive the formulae for velocity and acceleration in (planar) polar coordinates (r, φ), defined by

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

where (x, y) are the Cartesian coordinates of some inertial reference frame. At each point in the plane, we introduce the polar coordinate basis vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$. Unlike the Cartesian coordinate basis vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, the polar basis vectors are different at different points in the plane.

Fix the angular coordinate φ and vary the radial coordinate r; this parameterises a radial line. The derivative is

$$\frac{\partial x}{\partial r} = \cos\varphi$$
$$\frac{\partial y}{\partial r} = \sin\varphi$$

and therefore we define

$$\hat{\mathbf{r}} = (\cos\varphi\,, \sin\varphi)$$

as the radial basis vector at the point (r, φ). Notice that it is of unit length. Similarly, fix the radial coordinate r and vary the angular coordinate φ to obtain a parameterisation of a circle. The derivative is

$$\frac{\partial x}{\partial \varphi} = -r\sin\varphi$$
$$\frac{\partial y}{\partial \varphi} = r\cos\varphi.$$

and so we define

$$\hat{\boldsymbol{\varphi}} = (-\sin\varphi\,, \cos\varphi)$$

as the angular basis vector at the point (r, φ). Notice that it is of unit length and that

$$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\varphi}} = 0$$

everywhere; that is, the coordinate curves of the polar coordinate system always intersect orthogonally. Now, let $\mathbf{r}(t)$ be the geometric radius vector of a particle at time t. Let $\big(r(t), \varphi(t)\big)$ be the polar coordinates of the particle at this time. Then

$$\mathbf{r}(t) = r(t)\hat{\mathbf{r}}(t)$$

where $\hat{\mathbf{r}}(t)$ is the radial unit vector *at the point* $\big(r(t), \varphi(t)\big)$. Differentiation yields the velocity

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \dot{r}(t)\hat{\mathbf{r}}(t) + r(t)\dot{\hat{\mathbf{r}}}(t) = \\ &= \dot{r}(t)\hat{\mathbf{r}}(t) + r(t)(-\sin\varphi\,, \cos\varphi)\dot{\varphi}(t) = \\ &= \dot{r}(t)\hat{\mathbf{r}}(t) + r(t)\dot{\varphi}(t)\hat{\boldsymbol{\varphi}}(t)\end{aligned}$$

using (↑), (↑), and the chain rule. Differentiating again, we obtain the acceleration

$$\begin{aligned}\ddot{\mathbf{r}}(\mathrm{t}) &= \ddot{r}(t)\hat{\mathbf{r}}(t) + \dot{r}(t)\dot{\hat{\mathbf{r}}}(t) + \dot{r}(t)\dot{\varphi}(t)\hat{\boldsymbol{\varphi}}(t) + r(t)\ddot{\varphi}(t)\hat{\boldsymbol{\varphi}}(t) + r(t)\dot{\varphi}(t)\dot{\hat{\boldsymbol{\varphi}}}(t) = \\ &= \ddot{r}(t)\hat{\mathbf{r}}(t) + \dot{r}(t)(-\sin\varphi\,,\cos\varphi)\dot{\varphi}(t) + \dot{r}(t)\dot{\varphi}(t)\hat{\boldsymbol{\varphi}}(t) + r(t)\ddot{\varphi}(t)\hat{\boldsymbol{\varphi}}(t) + \\ &\qquad + r(t)\dot{\varphi}(t)(-\cos\varphi\,,-\sin\varphi)\dot{\varphi}(t) = \\ &= \ddot{r}(t)\hat{\mathbf{r}}(t) + \dot{r}(t)\dot{\varphi}(t)\hat{\boldsymbol{\varphi}}(t) + \dot{r}(t)\dot{\varphi}(t)\hat{\boldsymbol{\varphi}}(t) + r(t)\ddot{\varphi}(t)\hat{\boldsymbol{\varphi}}(t) - r(t)\dot{\varphi}(t)^2\hat{\mathbf{r}}(t) = \\ &= [\ddot{r}(t) - r(t)\dot{\varphi}(t)^2]\hat{\mathbf{r}}(t) + [2\dot{r}(t)\dot{\varphi}(t) + r(t)\ddot{\varphi}(t)]\hat{\boldsymbol{\varphi}}(t).\end{aligned}$$

1.8.4 The Simple Model

Initially, we will use a slightly simplified model of the solar system: We will assume that the Sun is *fixed* at the origin of an inertial coordinate system. Thus, assume that the Sun is located at $\mathbf{0}$ of our inertial frame $\mathcal{F}1 \in \mathcal{F}$, and let $\mathbf{r}$ be the position of a planet. Let M and m be the mass of the Sun and the planet, respectively. Then the net force on the planet is

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

in spherical coordinates (r, θ, φ).[18] The torque on the planet is

$$\boldsymbol{\tau} \overset{\text{def}}{=} \mathbf{r} \times \mathbf{F} = r\hat{\mathbf{r}} \times \left(-\frac{GMm}{r^2}\hat{\mathbf{r}}\right) = \mathbf{0}$$

and so its angular momentum is a constant of motion. What *is* the angular momentum? Well,

$$\mathbf{L} \overset{\text{def}}{=} \mathbf{r} \times \mathbf{p} = r\hat{\mathbf{r}} \times m\mathbf{v}.$$

Since this vector is constant, its direction is constant, and so the orbit is restricted to a plane, namely, the plane with normal direction $\mathbf{L}$. Indeed, since $\mathbf{L} = r\hat{\mathbf{r}} \times m\mathbf{v}$, $\mathbf{L} \perp \mathbf{v}$ at all times, and if $\mathbf{L}$ is a constant, then $\mathbf{v}$ has always to be orthogonal to the same direction, and all such vectors lie in the same plane. The magnitude of the angular momentum, which is also constant, is

$$L \coloneqq |\mathbf{L}| = rvm\sin\angle(\hat{\mathbf{r}}, \mathbf{v}).$$

Using (↑) and (↑) we can write this

$$L = |\mathbf{L}| = |r\hat{\mathbf{r}} \times m\mathbf{v}| = |r\hat{\mathbf{r}} \times m(\dot{r}\hat{\mathbf{r}}(t) + r\dot{\varphi}\hat{\boldsymbol{\varphi}})| = |r\hat{\mathbf{r}} \times mr\dot{\varphi}\hat{\boldsymbol{\varphi}}| = mr^2\dot{\varphi}.$$

It is customary to introduce the quantity

$$\mathbf{h} \coloneqq \frac{1}{m}\mathbf{L},$$

the angular momentum per unit mass. Its magnitude, which is constant, is

$$h \coloneqq |\mathbf{h}| = r^2\dot{\varphi}.$$

1.8.5 Are the Orbits Circular?

Simple [but not too accurate] astronomical observations seem to agree with the qualitative features of circular orbits around the sun for all the planets. We also know that Newton's laws do admit circular orbits. A planet orbiting the star in a circle of radius r requires the force to equal the centripetal force

$$F = \frac{mv^2}{r}$$

if its speed is v. Since the force is of magnitude

[18] This does *not* contradict our notation, according to which $r \coloneqq |\mathbf{r}|$. Indeed, the length $|\mathbf{r}|$ of the radius vector equals the radial coordinate r of the point at which the radius vector points.

$$F = \frac{GMm}{r^2}$$

theory requires

$$v = \sqrt{\frac{GM}{r}}.$$

That is, given a distance r from the star, the speed of the planet in its circular orbit needs to be precisely $\sqrt{GM/r}$, independently of the mass of the planet. Put differently, we have shown that Newton's laws admit circular orbits, and – in addition – that a planet that is put in orbit with the exactly right speed (given the distance to the star), *will* follow a circular orbit, indefinitely. But is *every* closed orbit necessarily a circle? It is not. The most general closed orbit is an *ellipse*, the circle being nothing but a special case of an ellipse. This is the content of Kepler's first law, as we will show next.

1.8.6 The General Orbit – Kepler's First Law

Newton's second law for the planet is

$$m\mathbf{a} = -\frac{GMm}{r^2}\hat{\mathbf{r}}.$$

Since we have already determined that the orbit is restricted to a single plane, we may – without loss of generality – choose the spherical coordinate system so that the orbit lies inside the $\theta = \pi/2$ plane. In this plane, the remaining spherical coordinates (r, φ) coincide with the (planar) polar coordinates.

Now, the acceleration vector $\mathbf{a}$ is a geometric vector, but its components differ between different coordinate systems. In polar coordinates, we have derived the acceleration

$$\mathbf{a} = (\ddot{r} - r\dot{\varphi}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\hat{\boldsymbol{\varphi}}$$

where the derivatives of r and φ are the derivatives of the planet's polar coordinates with respect to time and $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$ are the polar basis vectors *at the current position of the planet*. Hence, Newton's second law (↑) is equivalent to the system

$$\begin{aligned} \ddot{r} - r\dot{\varphi}^2 &= -\frac{GM}{r^2} \\ 2\dot{r}\dot{\varphi} + r\ddot{\varphi} &= 0 \end{aligned}$$

since $-\frac{GMm}{r^2}\hat{\mathbf{r}}$ is also a vector at the same point and expressed in the same (local) basis.

We will do two things now:

- Introduce a new radial coordinate, and
- introduce a new problem: Instead of finding $t \mapsto (r, \varphi)$, we wish to find $\varphi \mapsto r$.

The new radial coordinate is

$$u \coloneqq \frac{1}{r}$$

which implies, assuming $h \neq 0$ (so that $\dot{\varphi}$ is never zero),

$$h = \frac{\dot{\varphi}}{u^2}, \qquad r = \frac{1}{u}, \qquad \dot{r} = -\frac{\dot{u}}{u^2} = -\frac{1}{u^2}\frac{du}{d\varphi}\dot{\varphi} = -h\frac{du}{d\varphi},$$

and finally,

$$\ddot{r} = \frac{d}{dt}\dot{r} = \frac{d}{dt}\left(-h\frac{du}{d\varphi}\right) = -h\frac{d^2u}{d\varphi^2}\cdot\dot{\varphi}.$$

If we define

$$(\cdots)' \coloneqq \frac{d}{d\varphi}(\cdots), \qquad \forall(\cdots)$$

then we have transformed (↑) into

$$-h\dot{\varphi}u'' - u^{-1}\dot{\varphi}^2 = -GMu^2.$$

Divide by u^2 to obtain

$$-h^2u'' - h^2u = -GM,$$

that is, assuming $h \neq 0$,

$$u'' + u = \frac{GM}{h^2}.$$

This is an inhomogeneous constant-coefficient second-order linear ODE. The homogeneous solution is

$$u_H(\varphi) = A\sin\varphi + B\cos\varphi$$

and a particular solution to the full equation is

$$u_P(\varphi) = \frac{GM}{h^2}.$$

Thus, the full solution of (↑) is

$$u(\varphi) = A\sin\varphi + B\cos\varphi + \frac{GM}{h^2}.$$

We assume that $(A, B) \neq (0,0)$. Indeed, we already are familiar with the *circular* orbit. The Swedish *hjälpvinkelmetod* yields

$$u(\varphi) = C\cos(\varphi + \delta) + \frac{GM}{h^2}$$

for $C \coloneqq \sqrt{A^2 + B^2} > 0$ and some $\delta \in \mathbb{R}$. Obviously, we can choose the polar coordinate system in such a way that $\delta = 0$ and then the solution is

$$u(\varphi) = C\cos\varphi + \frac{GM}{h^2}.$$

Recalling that $u = \frac{1}{r}$, solving for r, and defining

$$\epsilon \coloneqq \frac{Ch^2}{GM} > 0, \qquad a \coloneqq \frac{\epsilon}{C(1-\epsilon^2)} = \frac{h^2}{GM(1-\epsilon^2)},$$

assuming $\epsilon \neq 1$, we obtain

$$r = \frac{a(1-\epsilon^2)}{\epsilon\cos\varphi + 1}.$$

1.8.6.1 Case 1

If

$$\epsilon \stackrel{\text{def}}{=} \frac{Ch^2}{GM} < 1$$

this is the polar equation of an ellipse with its right focus at the origin (where the Sun is!), semi-major axis $a > 0$, and eccentricity $\epsilon \in]0,1[$.

1.8.6.2 Case 2

If, instead,

$$\epsilon \stackrel{\text{def}}{=} \frac{Ch^2}{GM} > 1,$$

then we can define

$$\tilde{a} \coloneqq -a = -\frac{\epsilon}{C(1-\epsilon^2)} = \frac{\epsilon}{C(\epsilon^2-1)} > 0$$

to obtain

$$r = \frac{-\tilde{a}(1-\epsilon^2)}{\epsilon\cos\varphi + 1} = \frac{\tilde{a}(\epsilon^2-1)}{\epsilon\cos\varphi + 1}$$

which is the equation for the left branch of a *hyperbola* with left focus at the origin (where the Sun is), semimajor axis $\tilde{a} > 0$ and eccentricity $\epsilon > 1$.

1.8.6.3 Case 3

In case of the (exceedingly rare) coincident

$$\epsilon \stackrel{\text{def}}{=} \frac{Ch^2}{GM} = 1,$$

we obviously have to return to (↑), since in the next step (↑), we assumed $\epsilon \neq 1$. If $\epsilon = 1$, (↑) yields

$$r(\cos\varphi + 1) = \frac{1}{C}.$$

Recalling that $x = r\cos\varphi$ and $r = \sqrt{x^2+y^2}$, this may be written

$$x + \sqrt{x^2+y^2} = \frac{1}{C}$$

or

$$x = \frac{1}{2C} - \frac{C}{2}y^2$$

which clearly is a parabola. The origin (the location of the Sun) is a distance $1/2C$ to the left of the vertex of the parabola. This point is called, no surprise, the *focus* of the parabola.

1.8.6.4 Case 4

In the derivation above, we almost tacitly assumed that the constant $h \neq 0$. For completeness, we now investigate this (rather trivial) possibility. Indeed, if $h \stackrel{\text{def}}{=} r^2\dot{\varphi} \equiv 0$, motion is clearly purely radial, and Newton's law (↑) reduces to

$$\ddot{r} = -\frac{GM}{r^2},$$

which, of course, can be written

$$m\ddot{r} = -\frac{GMm}{r^2},$$

and it is obvious that the body is in free radial fall towards the Sun; the trajectory is thus a straight line before the body is engulfed by the Sun.

1.8.6.5 Summary

We have shown that the orbit of a massive particle in a gravitational field from a fixed star, such as a planet or a comet in the vicinity of the Sun, is either of the *conic sections*, that is, an ellipse, a hyperbola, or (rarely) a parabola. Of these, the only *closed* curve is the ellipse, and so it follows that the orbit of every planet around the Sun is an ellipse, and we have also shown that the Sun is at one of the foci. This is precisely the contents of Kepler's first law.

Distant comets approaching the solar system are 'slung' by the Sun, and we have shown that their orbits are almost certainly hyperbolae. The fact that the curvature of the hyperbola tends to zero as we move away from its point of symmetry is clearly related to the fact that, far from the Sun, (almost) no forces act on the comet, and so, according to Newton's first law of motion, it travels (almost) along a straight line.

1.8.7 Kepler's Second Law

We have found that the angular momentum

$$L = mr^2\dot{\varphi}$$

is a constant of motion, and, in fact, this statement is equivalent to Kepler's second law. Indeed, the area swept out by the radius vector during an (infinitesimal) time dt is equal to the circular sector area

$$dA = \frac{1}{2}r^2 d\varphi$$

where $d\varphi$ is the corresponding change in the polar coordinate. But

$$dA = \frac{1}{2}r^2 d\varphi = \frac{1}{2}r^2\frac{d\varphi}{dt}\cdot dt = \frac{1}{2}r^2\dot{\varphi}dt = \frac{1}{2}hdt$$

where $h \overset{\text{def}}{=} L/m$ is the (constant) angular momentum per unit mass, which implies

$$\frac{dA}{dt} = \frac{1}{2}h = \text{constant}.$$

That is, the rate of sweeping out area is constant in time, which is precisely the second law of planetary motion.

1.8.8 Kepler's Third Law

The orbit of a planet is an ellipse

$$r = \frac{a(1-\epsilon^2)}{\epsilon\cos\varphi + 1}$$

where a is the semi-major axis length and ϵ is the eccentricity. Thus the semi-minor axis length is $b = a\sqrt{1-\epsilon^2}$ and therefore the area of the ellipse is

$$A = ab\pi = a^2\pi\sqrt{1-\epsilon^2}.$$

Kepler's second law (↑) then implies that the orbital period of the planet

$$T \coloneqq \frac{A}{dA/dt} = \frac{2a^2\pi\sqrt{1-\epsilon^2}}{h}.$$

But according to (↑),

$$a(1-\epsilon^2) = \frac{h^2}{GM}$$

which is a link between geometrical properties of the ellipse (the LHS) and dynamical properties of the planetary orbit (the RHS); in particular,

$$h = \sqrt{GMa(1-\epsilon^2)};$$

thus (↑) can be written

$$T = \frac{2a^2\pi\sqrt{1-\epsilon^2}}{\sqrt{GMa(1-\epsilon^2)}} = \frac{2\pi}{\sqrt{GM}} \cdot a^{3/2}.$$

That is, $T \propto a^{3/2}$ and the constant of proportionality, $\frac{2\pi}{\sqrt{GM}}$, is a property of the star (or the 'planetary system'), and is therefore the same for any planet in the system, and this is precisely the contents of the third law of planetary motion.

1.8.9 The Validity of the Simple Model

We have considered a somewhat simplified model of the Sun–planet system. We have assumed that the Sun is *fixed* at the origin of our inertial reference frame, but that is clearly not the case. Indeed, since the planet is affecting the Sun with the force of gravity (equal in magnitude to the force by which the Sun affects the planet), the Sun is accelerating, and so there cannot exist *any* inertial frame relative to which the Sun is always at rest. Notice, however, that the motion of the Sun is extremely small compared with the motion of the planet, since the Sun is very much heavier. Indeed, the ratio between the mass of the Sun and the mass of the Earth is about 333 000. The heaviest planet in the solar system is Jupiter. The Sun–Jupiter mass ratio is about 1 050. Hence, even in this case, it is a fairly decent approximation to consider the Sun as being fixed at the origin. At the other extreme, the mass ratio between the Sun and a comet or asteroid is huge, and the approximation is essentially without error; for instance, the ratio between the mass of the Sun and the mass of Halley's comet is almost 10^{16}.

We have also neglected the gravitational influence of the other planets (and other bodies) orbiting the Sun; of course, these will influence the Sun–planet system. However, again, the mass of a planet is generally negligible compared to that of the Sun, and so the Sun is barely affected. Also, the influence on the Earth (say) from the other planets in the solar system is almost always negligible compared to gravitational attraction from the Sun for the same reason.

Still, one might wonder if there is a way to improve the model, and, indeed, there is. Since the solar system is very isolated in the galaxy, the gravitational field *due to the rest of the galaxy* is almost constant inside the solar system. Hence, any particle in free fall in this region of space, were it not for the solar system, would serve as the origin of an inertial frame to an outstanding approximation. As we have shown in previous section, such a 'particle' is the centre of mass of the solar system. In other words, the CM frame of the solar system is an almost perfect inertial

frame. This frame should be used for a more advanced treatment. In particular, this would make it possible to study systems of two bodies, gravitationally bound to each other, where their masses are comparable, e.g. binary star systems, or big moons of small planets (considering only the planet–moon system).

Let us investigate the centre of mass of a two-body system. Hence, let A and B be two massive bodies, where the masses may be of the same order of magnitude. If their positions are $\mathbf{r}_A$ and $\mathbf{r}_B$, relative to some frame of reference, and their masses are m_A and m_B, respectively, then the CM is located at

$$\mathbf{R} = \frac{1}{M}(m_A\mathbf{r}_A + m_B\mathbf{r}_B).$$

where $M \coloneqq m_A + m_B$ is the total mass. Hence,

$$\mathbf{R} = \frac{1}{M}(m_A\mathbf{r}_A + m_B\mathbf{r}_B) - \mathbf{r}_A + \mathbf{r}_A = \left(\frac{m_A}{M} - 1\right)\mathbf{r}_A + \frac{m_B}{M}\mathbf{r}_B + \mathbf{r}_A = \left(\frac{-m_B}{M}\right)\mathbf{r}_A + \frac{m_B}{M}\mathbf{r}_B + \mathbf{r}_A =$$
$$= \mathbf{r}_A + \frac{m_B}{M}(\mathbf{r}_B - \mathbf{r}_A).$$

The geometric interpretation of this is obvious: you get to the CM ($\mathbf{R}$) by going to the first body ($\mathbf{r}_A$) and then you go m_B/M of the displacement from $\mathbf{r}_A$ to $\mathbf{r}_B$. Hence, in the limit $m_B/M \to 0$ (essentially all the mass belongs to A), $\mathbf{R} \to \mathbf{r}_A$, while in the limit $m_B/M \to 1$ (essentially all the mass belongs to B), $\mathbf{R} \to \mathbf{r}_B$. Let us consider a few actual figures: let A be the Sun and B be the Earth, Jupiter, and Halley's comet, respectively. Then the values for m_B/M are as follows.

B	Earth	Jupiter	Halley's comet
m_B/M	$3.0 \cdot 10^{-6}$	$9.5 \cdot 10^{-4}$	$1.1 \cdot 10^{-16}$

The actual distance between A and the CM is, from (↑),

$$|\mathbf{R} - \mathbf{r}_A| = \frac{m_B}{M}|\mathbf{r}_B - \mathbf{r}_A|.$$

Assume that the distance between the Sun and the Earth, Jupiter, and Halley's comet is 1 AU, 5 AU, and 33 AU, respectively. Then the corresponding normalised distances are given below; the values are normalised in terms of the solar radius $r_\odot$.

B	Earth	Jupiter	Halley's comet
$\lvert\mathbf{R} - \mathbf{r}_A\rvert/r_\odot$	0.065 %	100 %	$7.8 \cdot 10^{-11}$ %

Clearly, the CM of both the Sun–Earth and the Sun–Halley system lies almost at the core of the Sun, while the CM of the Sun–Jupiter system lies almost precisely at the surface of the Sun.

1.9 Classical Gravity – Gaussian Formalism

The foundation of classical gravity is Newton's law of universal gravitation,

$$\mathbf{F} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

introduced in previous sections. In this section we will derive Gauss' law for gravity. This is actually a very easy task owing to the general machinery of vector calculus, but it consists of a fair number of steps, and we will investigate each one of them in detail. Let's start with the core of the topic:

Lemma

Assume that the gravitational field is

$$\mathbf{G}(\mathbf{x}) = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

(that is, there is a particle of (active gravitational) mass M at the origin). Let S be a sphere of radius $r > 0$ centred at the origin. Then[19]

$$\oiint_S \mathbf{G}\cdot d\mathbf{A} = -4\pi GM$$

Proof

$$\oiint_S \mathbf{G}\cdot d\mathbf{A} = \oiint_S \left(-\frac{GM}{r^2}\hat{\mathbf{r}}\right)\cdot(r^2\sin\theta\,\hat{\mathbf{r}})d\theta d\varphi = \oiint_S (-GM\sin\theta)d\theta d\varphi =$$
$$= -GM\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\varphi = -4\pi GM.$$

■

Since the gravitational field is conservative (it has a potential), it is irrotational. But it is also of zero divergence.

Lemma

The gravitational field

$$\mathbf{G}(\mathbf{x}) = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

is divergence-free, that, is, $\nabla\cdot\mathbf{G} = 0$ everywhere except at the origin.

Proof

$$\nabla\cdot\mathbf{G} = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left[r^2\sin\theta\cdot\left(-\frac{GM}{r^2}\right)\right] + \frac{\partial}{\partial\theta}[r\sin\theta\cdot 0] + \frac{\partial}{\partial\varphi}[r\cdot 0]\right) = 0$$

[19] When talking about the flux through a *closed* surface we always mean the flux from the interior to the exterior of the surface, unless the opposite is explicitly stated.

because the r-dependance in the field component $-GM/r^2$ is precisely compensated by the scale factor $r^2 \sin\theta$ of the coordinate system. ■

Lemma

Let

$$\mathbf{G}(\mathbf{x}) = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

be the gravitational field. Then

$$\oiint_S \mathbf{G} \cdot d\mathbf{A} = -4\pi GM$$

for *any* closed surface S enclosing the origin.

Proof

Let $\delta \coloneqq \min_{\mathbf{r}\in S}|\mathbf{r}|$ be the distance between the origin and S. Introduce the sphere S' of radius $\delta/2$ centred at the origin, and let D denote the bounded region in space the boundary of which is $\partial D = S \cup S'$.

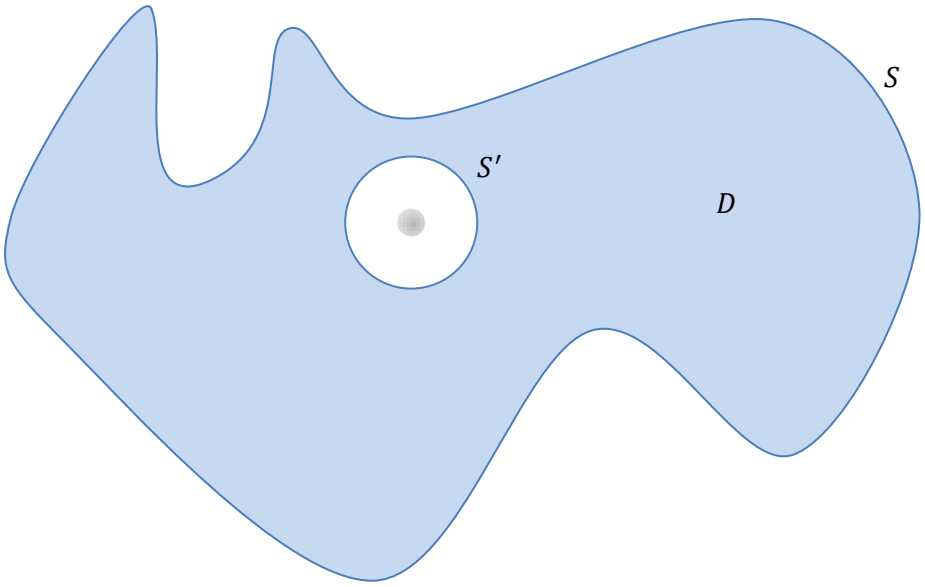


Figure 20. A bounded region D.

By Lemma NN,

$$\oiint_{S'} \mathbf{G} \cdot d\mathbf{A} = -4\pi GM$$

and, applying the divergence theorem to D and ∂D, we find

$$\oiint_{\partial D} \mathbf{G} \cdot d\mathbf{A} = \oiint_{\substack{S',\text{ out}\\ \text{from } D}} \mathbf{G} \cdot d\mathbf{A} + \oiint_S \mathbf{G} \cdot d\mathbf{A} = \iiint_D \nabla \cdot \mathbf{G} dV = 0 \Rightarrow \oiint_S \mathbf{G} \cdot d\mathbf{A} = -4\pi GM$$

where $\oiint_{\substack{S',\text{ out}\\ \text{from } D}} \mathbf{G} \cdot d\mathbf{A} = 4\pi GM$ is the flux through S' in the direction *out of* D, that is, *towards* the origin. ■

Lemma

Let

$$\mathbf{G}(\mathbf{x}) = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

be the gravitational field. Then

$$\oiint_S \mathbf{G} \cdot d\mathbf{A} = 0$$

for *any* closed surface S *not* enclosing the origin.

Proof

Let D be the volume enclosed by S, so that $S = \partial D$. Then

$$\oiint_S \mathbf{G} \cdot d\mathbf{A} = \iiint_D \nabla \cdot \mathbf{G} dV = 0.$$

■

Lemma

Let S be a closed surface, the *interior* of which contains a particle of mass M, producing a gravitational field $\mathbf{G}$. Then $\oiint_S \mathbf{G} \cdot d\mathbf{A} = -4\pi GM$.

Proof

Choose a coordinate system in which the particle is at the origin and apply Lemma NN. ■

Lemma

Let S be a closed surface, the *exterior* of which contains a particle of mass M, producing a gravitational field $\mathbf{G}$. Then $\oiint_S \mathbf{G} \cdot d\mathbf{A} = 0$.

Proof

Choose a coordinate system in which the particle is at the origin and apply Lemma NN. ■

Lemma

Let S be a closed surface, and consider a system of N particles (index set X) with positions $\mathbf{r}_i$ and masses m_i. Let $\chi_i = [\mathbf{r}_i \in \mathrm{int}(S)]$. Let $\mathbf{G}$ be the *total* gravitational field due to all N particles. Then

$$\oiint_S \mathbf{G} \cdot d\mathbf{A} = -4\pi GM$$

where

$$M \coloneqq \sum_{i \in X} m_i \chi_i$$

is the total mass *inside* S.

Proof

The gravitational field at $\mathbf{r}$ from the ith particle alone is

$$\mathbf{G}_i(\mathbf{r}) \coloneqq -\frac{Gm_i}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i).$$

Since you add forces to get the net force, the net gravitational field is

$$\mathbf{G} = \sum_{i\in X} \mathbf{G}_i.$$

Then the flux

$$\oiint_S \mathbf{G}\cdot d\mathbf{A} = \oiint_S \left(\sum_{i\in X}\mathbf{G}_i\right)\cdot d\mathbf{A} = \oiint_S \left(\sum_{i\in X}\mathbf{G}_i\cdot d\mathbf{A}\right) = \sum_{i\in X}\left(\oiint_S \mathbf{G}_i\cdot d\mathbf{A}\right).$$

Now, by Lemmas NN and NN,

$$\oiint_S \mathbf{G}_i\cdot d\mathbf{A} = -4\pi G m_i \chi_i.$$

Thus

$$\oiint_S \mathbf{G}\cdot d\mathbf{A} = \sum_{i\in X}(-4\pi G m_i\chi_i) = -4\pi GM.$$

■

We have therefore shown

Proposition (Gauss' Law for Gravity)

Let S be a closed surface encompassing a total mass M. If $\mathbf{G}$ is the geometric net gravitational field in space, then

$$\oiint_S \mathbf{G}\cdot d\mathbf{A} = -4\pi GM.$$

If we have instead a continuous distribution of mass, given by a mass density function ρ, then we obtain

Corollary

$$\nabla\cdot\mathbf{E} = -4\pi G\rho.$$

Proof

Let D be any bounded region with boundary $S = \partial D$. Then Gauss' law

$$\oiint_S \mathbf{G}\cdot d\mathbf{A} = \iiint_D \nabla\cdot\mathbf{G}\,dV$$

but

$$\oiint_S \mathbf{G} \cdot d\mathbf{A} = -4\pi GM = -4\pi G \iiint_D \rho dV.$$

Thus, since this has to hold for every bounded region D,

$$\nabla \cdot \mathbf{G} = -4\pi G\rho.$$

■

1.9.1 Applications of Gauss' Law for Gravity

Proposition

Consider, centred at the origin, a spherically symmetric body A of total mass M and radius R. Outside the body (at $r > R$), the gravitational field from A is identical to the field that a point particle of mass M located at origin would produce.

Proof

Let S_r be a sphere of radius $r > R$ centred at the origin. Because the body is spherically symmetric, it is obvious that the actual gravitational field is $\mathbf{G}(\mathbf{x}) = G(r)\hat{\mathbf{r}}$ for some function G. Now Gauss' law for gravity states

$$\oiint_{S_r} \mathbf{G} \cdot d\mathbf{A} = -4\pi GM$$

while

$$\oiint_{S_r} \mathbf{G} \cdot d\mathbf{A} = \oiint_{S_r} G(r)\hat{\mathbf{r}} \cdot r^2 \sin\theta\, \hat{\mathbf{r}} d\theta d\varphi = G(r)r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi = 4\pi G(r)r^2.$$

Combining these results we find

$$-4\pi GM = 4\pi G(r)r^2$$

and so

$$G(r) = -\frac{GM}{r^2}$$

yielding

$$\mathbf{G}(\mathbf{x}) = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

■

Consequently, as long as the body is spherically symmetric and has mass M, the gravitational field outside the body is independent of the internal structure of the body. For instance, a point particle of mass M produce the same external field as a huge star of total mass M, as does a thin (and hollow) spherical shell of total mass M.

Proposition

Inside a spherically symmetric shell of radius R, the net gravitational field due to the shell vanishes.

Proof

Same as last proposition but with $r < R$ and $\oiint_{S_r} \mathbf{G} \cdot d\mathbf{A} = 0$. ■

Consider now a spherically symmetric ball. It is obvious (from symmetry) that the net gravitational field from the ball vanishes at the centre of the ball. Moreover, if we know the radial distribution of mass (the radial density function), we can compute the field at every point. The simplest case is given below.

Proposition

Consider a homogeneous ball of density ρ and radius R. The gravitational field tends linearly to zero as one approaches the centre of the ball along a radius.

Proof

We wish to find the gravitational field $\mathbf{G}$ inside the ball. From symmetry, it is clear that $\mathbf{G}(\mathbf{x}) = G(r)\hat{\mathbf{r}}$. Let S_r be a sphere of radius $r \leq R$ concentric with the ball. Inside S_r, the total mass is

$$M_r = \frac{4}{3}\pi r^3 \rho.$$

Therefore, Gauss' law for gravity reads

$$\oiint_{S_r} \mathbf{G} \cdot d\mathbf{A} = -\frac{16}{3}\pi^2 G\rho r^3$$

while

$$\oiint_{S_r} \mathbf{G} \cdot d\mathbf{A} = \oiint_{S_r} G(r)\hat{\mathbf{r}} \cdot r^2 \sin\theta\, \hat{\mathbf{r}} d\theta d\varphi = G(r)r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi = 4\pi G(r)r^2.$$

Thus

$$-\frac{16}{3}\pi^2 G\rho r^3 = 4\pi G(r)r^2$$

yielding

$$G(r) = -\frac{4}{3}\pi G\rho r$$

and

$$\mathbf{G}(\mathbf{x}) = -\frac{4}{3}\pi G\rho r\hat{\mathbf{r}}.$$

Clearly $G \propto r$. ■

Occasionally the last three propositions are referred to as the *shell theorems*. Needless to say, it requires quite a lot of work to prove them using only the usual form (↑) of Newton's law of universal gravitation (and not the Gaussian formalism).

Combining Propositions NN-M, we have that a ball of radius R, constant density ρ, and total mass $M = \frac{4}{3}\pi R^3\rho$ produces the gravitational field $\mathbf{G}(r) = -\frac{4}{3}\pi G\rho r\hat{\mathbf{r}} = -\frac{GMr}{R^3}\hat{\mathbf{r}}$ in its interior ($r < R$) and the field $\mathbf{G}(r) = -\frac{GM}{r^2}\hat{\mathbf{r}}$ in its exterior ($r > R$). Thus, the potential is

$$\Phi(r) = \begin{cases} \frac{GM}{2R^3}r^2 - \frac{3GM}{2R} & r \leq R \\ -\frac{GM}{r} & r > R \end{cases}$$

where we have shifted the interior potential by an amount $-\frac{3GM}{2R}$ in order to match it to the exterior potential. We chose to shift the interior potential and not the exterior potential because we are fond of the conventional behaviour $\phi(r) \to 0$ as $r \to \infty$.

The Sun is most definitely not of constant density, but to get an idea about the potential, we will use the mass and radius of the sun as M and R, respectively, to plot $r \mapsto \phi(r)$.

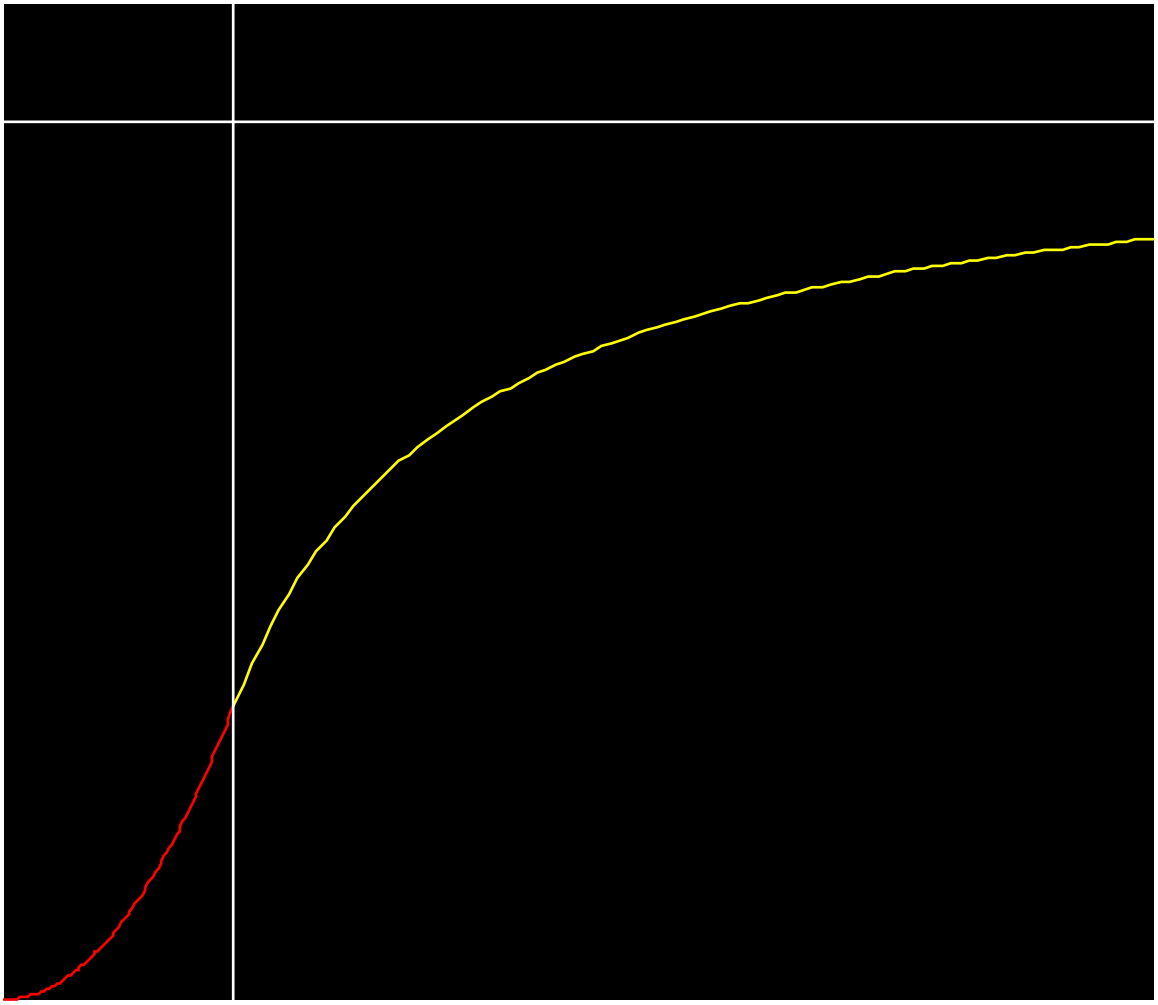


Figure 21. The gravitational potential inside and outside a homogeneous and spherically symmetric body.

Notice that the slope tends to zero both as $r \to 0^+$ and as $r \to \infty$, as expected. The highest slope (that is, the strongest field) is found near the surface $r = R$. Of course, we can also find the potential along a full straight line through the origin. Indeed, we only need to construct the even extension of Φ. Doing so, we obtain

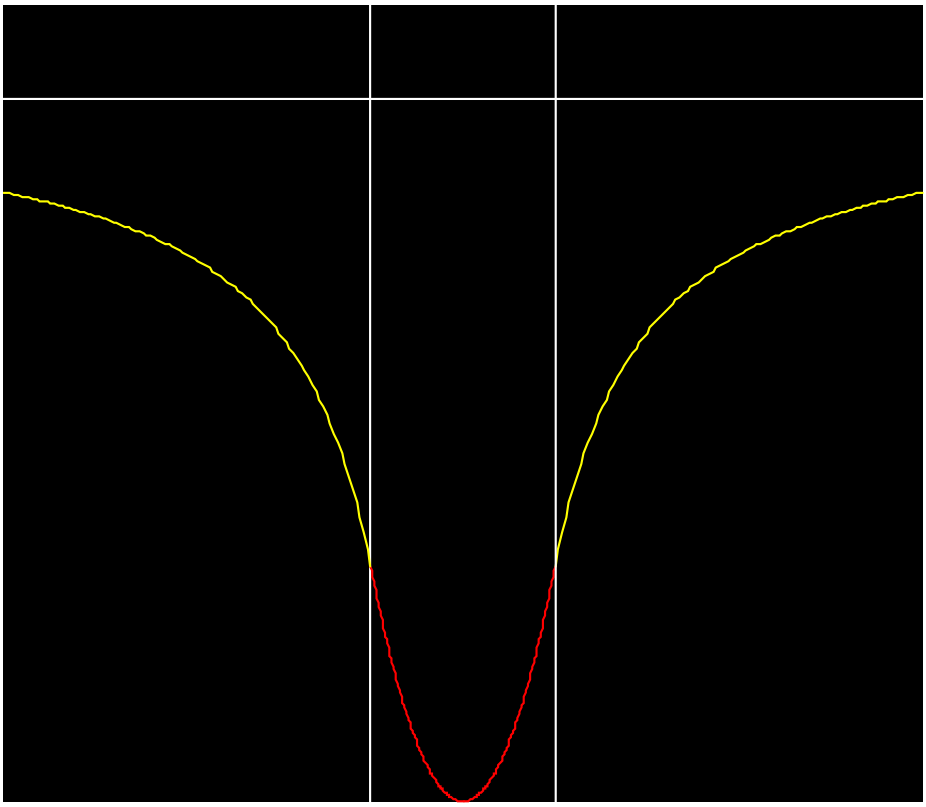


Figure 22. The gravitational potential inside and outside a homogeneous and spherically symmetric body.

Finally, replace the solid ball by a thin, hollow shell of the same mass and radius. Then the interior field is zero, and so the potential is constant:

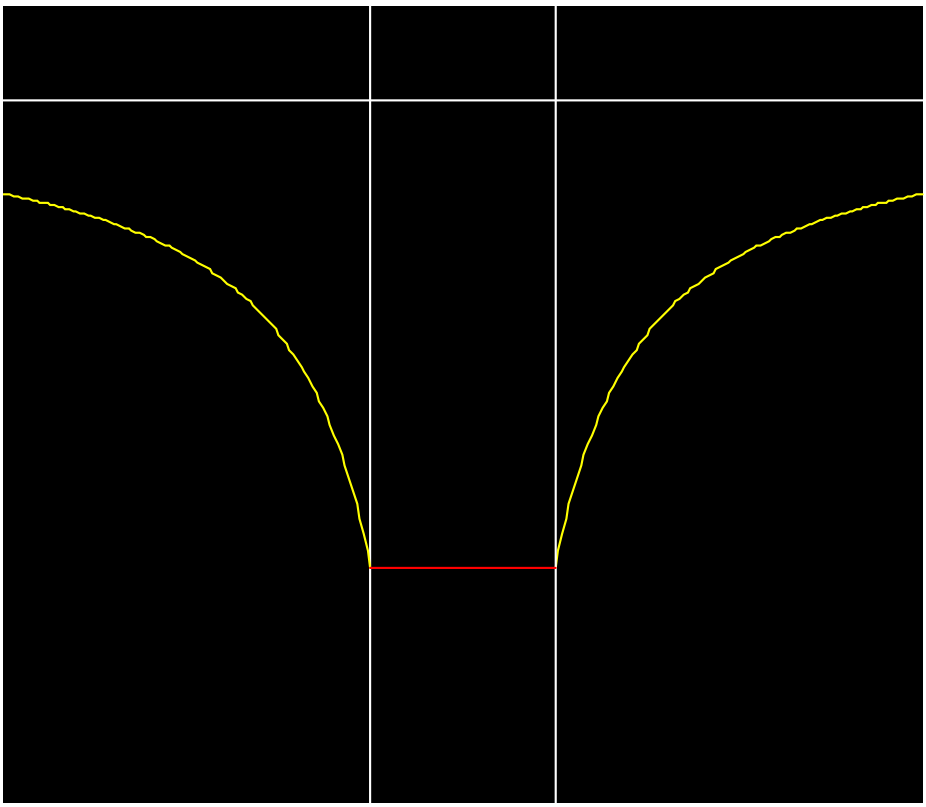


Figure 23. The gravitational potential inside and outside a homogeneous and spherically symmetric thin shell.

Of course, all these images should be compared to the graph of the potential form a point mass (of the same mass) at the origin:

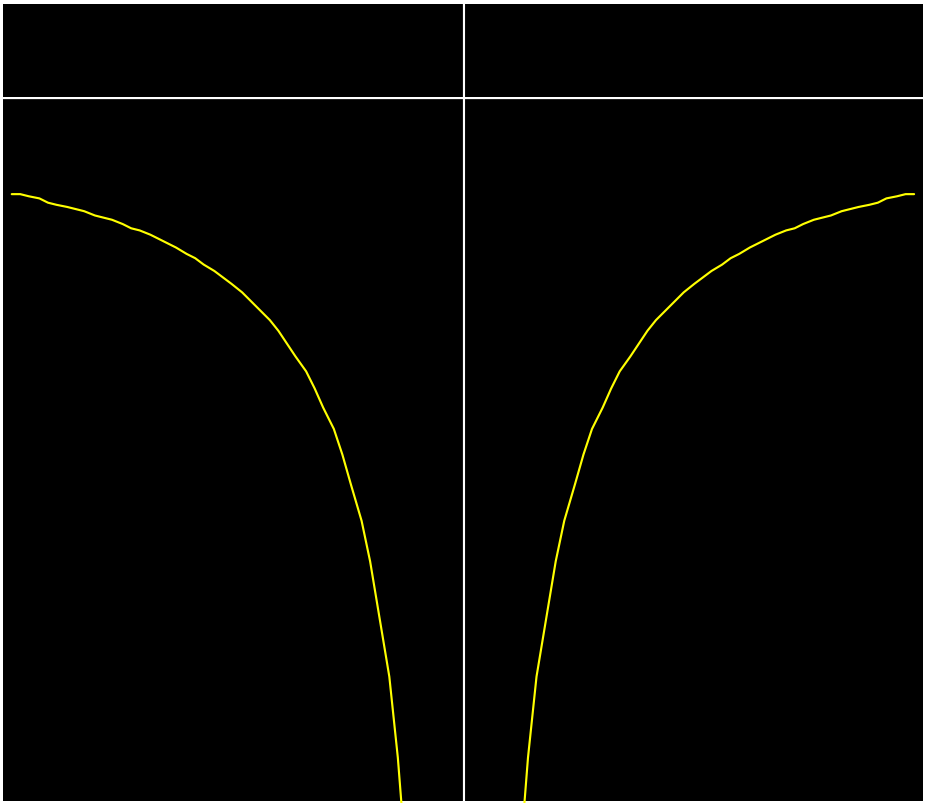


Figure 24. The gravitational potential from a point mass at the origin.

2 Classical Electromagnetism

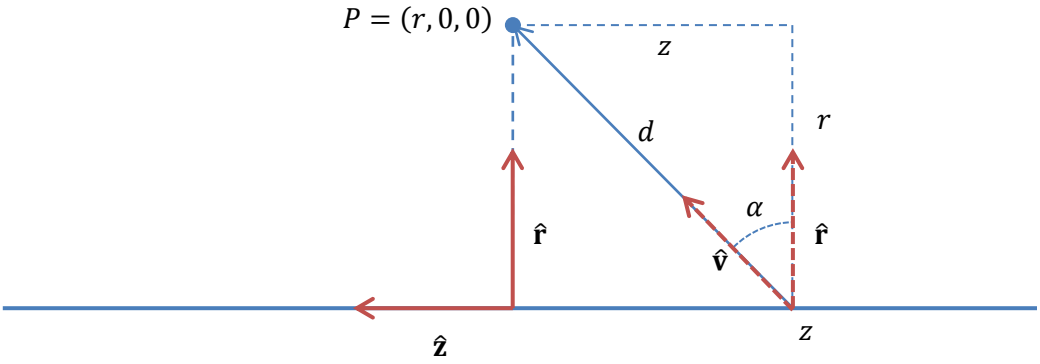


Figure 25. A diagram that we will use in Section 2.3.1 to find the electrostatic field from an infinitely long, uniformly charged wire, using only the classical Coulomb law. In this chapter, we will also introduce the Maxwell equations, and use them to 'rediscover' one of the most fascinating breakthroughs in the history of science, namely, the fact that light is an electromagnetic wave.

2.1 The Four Classical Laws

We will now review the four major results of classic electromagnetic theory. In addition to Coulomb's law, which describes the *electric* field **E** due to (static) charged particles, we are familiar with the Biot–Savart law that gives the *magnetic* field due to a (stationary) *current* of charged particles. If the current I [unit C/s] flows through a 'wire' (curve) Γ, then the magnetic field at $\mathbf{x}$ is

$$\mathbf{B}(\mathbf{x}) = \int_\Gamma \frac{\mu_0 I d\mathbf{l} \times \hat{\mathbf{v}}(\mathbf{x}')}{4\pi r(\mathbf{x}')^2}$$

where μ_0 is the magnetic constant[20], $\mathbf{x}' \in \Gamma$ is the current point on the curve, $d\mathbf{l}$ is the vector line element, $r(\mathbf{x}') \coloneqq |\mathbf{x} - \mathbf{x}'|$ is the distance from $\mathbf{x}'$ to $\mathbf{x}$, and $\hat{\mathbf{v}}(\mathbf{x}')$ is a unit vector pointing from $\mathbf{x}'$ to $\mathbf{x}$. The Biot–Savart law implies the Ampère's circuital law

$$\oint_C \mathbf{B} d\mathbf{l} = \mu_0 I$$

where I is the charge current flowing through the closed loop C [that is, flowing through any surface the boundary of which is C]. Notice that magnetic forces arise when charges are in motion only. Later on, we will see that the special theory of relativity actually describes the magnetic field as a relativistic correction of the electrostatic field. In other words, electric and magnetic forces are merely two aspects of the same physical phenomenon, the *electromagnetic* field. Hence, electric and magnetic forces can be *unified* in a single electromagnetic theory. This is why we talk about the *four* forces of nature, and not the *five* forces that would result if we replaced 'electromagnetic' with 'electric' and 'magnetic' as two separate entities.

In fact, unification of the forces of nature into more fundamental theories is one of the most ultimate goals of modern theoretical physics; this far the greatest achievement in this field is the unification of the electromagnetic forces with the weak nuclear force, the so-called *electroweak* unification. This, however, goes far beyond 'simple' relativity theory, so we will not discuss that topic, but settle for a description of the beautiful electromagnetic unification.

We have now formulae describing the *formation* of the electric and magnetic fields. The third ingredient of electrodynamics is *Faraday's law of induction*, which quantifies the phenomenon of electromagnetic induction, that is, the generation of an electric potential due to a time-varying magnetic field. The standard expression for this law is

$$\oint_{\partial S} \mathbf{E} d\mathbf{l} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} d\mathbf{A}$$

where S is any surface and ∂S its boundary. The surface integral $\iint_S \mathbf{B} d\mathbf{A}$ of the magnetic field is called the magnetic *flux*, and therefore the law states that the electromotive force [which is a potential, not a force] $\oint_{\partial S} \mathbf{E} d\mathbf{l}$ around a closed circuit is proportional to the time rate of change of the magnetic flux through the circuit.

The fourth and final law of electrodynamics is the law stating the absence of 'magnetic monopoles', namely

$$\oiint_{\partial V} \mathbf{B} d\mathbf{A} = 0,$$

[20] In SI units, $\mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am.

where $S = \partial V$ is any closed surface in space [being the boundary of some volume V]. This law is also a consequence of the Biot–Savart law.

2.2 Maxwell's Equations

The laws of electromagnetism are normally stated as the *Maxwell equations*:

Original law	Differential form	Integral form	
Coulomb's law	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\oiint_{\partial V} \mathbf{E} d\mathbf{A} = \frac{Q}{\epsilon_0}$	(M1)
Ampère's law	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_{\partial S} \mathbf{B} d\mathbf{l} = \mu_0 \iint_S \mathbf{J} d\mathbf{A} + \mu_0 \epsilon_0 \iint_S \frac{\partial \mathbf{E}}{\partial t} d\mathbf{A}$	(M2)
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} d\mathbf{l} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} d\mathbf{A}$	(M3)
No magnetic monopoles	$\nabla \cdot \mathbf{B} = 0$	$\oiint_{\partial V} \mathbf{B} d\mathbf{A} = 0$	(M4)

In this table, V is any volume and S is any non-closed surface. Consequently, ∂V is a closed surface and ∂S is a closed curve. When formulated in this way, the Coulomb law (M1) is often called 'Gauss' law', and the 'no monopoles' law (M4) is called the 'Gauss' law for magnetism'. Notice that the differential forms and the integral forms of these two laws are trivially converted to and from each other by means of the divergence theorem, or Gauss' law of vector calculus. On the other hand, the differential and integral forms of the Ampère and Faraday laws are trivially converted to and from each other by means of Stoke's theorem. Thus, the set of four 'differential form' equations are equivalent to the set of the four 'integral form' equations. Further, the Maxwell equations (in either form) are equivalent to the classical four theorems, with one notable exception, as we will point out shortly.

We will now investigate the Maxwell equations one by one. The first one, (M1), relates the *charge density* scalar field ρ [unit $\mathrm{C/m^3}$] to the divergence of the electric field $\mathbf{E}$. It follows from the notation used in the table that $Q = \iiint_V \rho dV$ is the total charge enclosed by ∂V. (M1) is equivalent to the old Coulomb inverse square law in precisely the same way that Gauss' law for gravity (↑) is related to Newton's law of universal gravitation (↑). Just as in the case of gravity, the Gaussian form Coulomb's law is superior in situations where symmetry is present

The second Maxwell equation, (M2), relates the curl of the magnetic field to the charge current density field $\mathbf{J}$ [unit $\mathrm{C/s{\cdot}m^2 = A/m^2}$] and the time derivative of the electric field. With the restriction $\partial \mathbf{E}/\partial t = \mathbf{0}$, as is the case if a constant current is flowing in an infinitely long, straight conductor, this is equivalent to the old Ampère's circuital law. The additional term $\mu_0 \epsilon_0 \, \partial \mathbf{E}/\partial t$ is called *Maxwell's correction*, and its necessity is displayed in most courses in elementary electromagnetics. Notice that $\iint_S \mathbf{J} d\mathbf{A} = I$ is the current [unit $C/s = A$] flowing through the surface S.

The third Maxwell equation, (M3), is trivially seen to be Faraday's law of electromagnetic induction, and the fourth Maxwell equation, (M4), states that there is no 'magnetic charge' like the 'electric charge'.

2.2.1 The Bridge between Mathematics and Physics

The Maxwell equations completely describe the electromagnetic field. But what *is* 'the electromagnetic field'? Indeed, the electric and magnetic fields are mere mathematical constructs that assign vectors to each point in space: what are these vectors? The physical interpretations of these fields, or the *definitions* of the fields, are as follows:

- The electric field **E** is the unique vector field satisfying $\mathbf{F} = q\mathbf{E}$ where **F** is the (physically measurable) *force* on a test particle with charge $q \neq 0$, which is at rest with respect to the system of coordinates.
- The magnetic field **B** is the unique vector field satisfying $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$ where **F** is the (physically measurable) *force* on a test particle with charge $q \neq 0$ and velocity $\mathbf{v} \neq \mathbf{0}$ with respect to the system of coordinates, and where **E** is the (now well-defined) electric field.

2.3 Examples of Electromagnetic Fields

We will derive some simple examples of electromagnetic fields.

2.3.1 The Electric Field from a Uniform and Infinite Charged Wire

As our first example, we will derive the expression for the electrostatic field outside an infinitely long wire with a constant charge density [unit C/m]. First we will do this from first principles, using only Coulomb's law. Then we will redo the derivation using Gauss' law. This way we can both confirm the latter and better appreciate the benefit of the latter in systems with symmetry.

2.3.1.1 Approach 1: Coulomb's Law

Introduce a cylindrical coordinate system such that the wire is situated along the z axis. From symmetry reasons, the electric field cannot depend on z or φ; nor can the electric field vector at any given point have a non-zero projection onto $\hat{\mathbf{z}}$ or $\hat{\boldsymbol{\varphi}}$. Thus $\mathbf{E} = E(r)\hat{\mathbf{r}}$. Now consider the point $(r, \varphi, z) = (r, 0, 0)$. By our symmetry considerations, it suffices to compute the electric field at this point (which depends on r only).

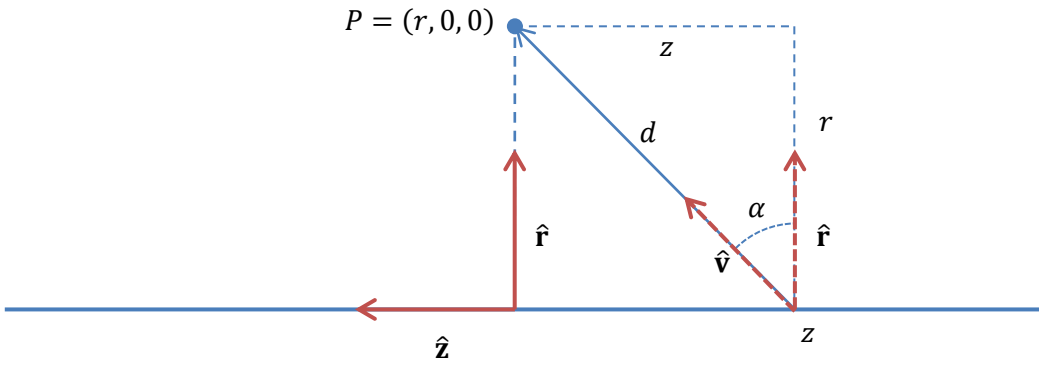


Figure 26. An infinite, uniformly charged metal wire.

Consider a small part dz of the wire located at z (between z and $z + dz$, say). The distance between this part and the field point P is $d(z) \coloneqq \sqrt{z^2 + r^2}$, and so this part of the wire will contribute with an electric field

$$d\mathbf{E}(z) = \frac{\rho dz}{4\pi\epsilon_0 d(z)^2}\hat{\mathbf{v}}(z)$$

where $\hat{\mathbf{v}}(z)$ is the direction from the part dz to the field point P. But since we have already determined that the net electric field at P (or anywhere) has only a radial component, it clearly suffices to consider only the radial projection of each $d\mathbf{E}(z)$. This is

$$dE_r(z) = \frac{\rho dz}{4\pi\epsilon_0 d(z)^2}\hat{\mathbf{v}}(z)\cdot\hat{\mathbf{r}} = \frac{\rho dz}{4\pi\epsilon_0 d(z)^2}\cos\alpha.$$

But the angle

$$\alpha \coloneqq \angle(\hat{\mathbf{v}}, \hat{\mathbf{r}}) = \arctan\frac{z}{r}$$

and, therefore,

$$\cos\alpha = \cos\arctan\frac{z}{r} = \frac{1}{\sqrt{1 + \frac{z^2}{r^2}}} = \frac{r}{\sqrt{r^2 + z^2}}$$

and

$$dE_r(z) = \frac{r\rho dz}{4\pi\epsilon_0(z^2+r^2)^{3/2}}.$$

Thus, the total field experienced at P is

$$E_r(r) = \int_{\mathbb{R}} dE_r(z) = \int_{\mathbb{R}} \frac{r\rho dz}{4\pi\epsilon_0(z^2+r^2)^{3/2}} = \frac{\rho}{2\pi\epsilon_0 r},$$

that is,

$$\mathbf{E} = \frac{\rho}{2\pi\epsilon_0 r}\hat{\mathbf{r}}.$$

2.3.1.2 Approach 2: Gauss' Law

The derivation of (↑) is far simpler using Gauss' law. By symmetry (again), the electric field $\mathbf{E}(r) = E(r)\hat{\mathbf{r}}$ does only depend on the radial coordinate, and at any point, the field vector can only point along the radial direction. Now, consider a solid cylinder D of radius R and height h with the z axis as its symmetry axis. The total encompassed charge (inside S) is $Q \coloneqq \rho h$. Thus Gauss' law states

$$\oiint_{\partial D} \mathbf{E}d\mathbf{A} = \frac{Q}{\epsilon_0} = \frac{\rho h}{\epsilon_0}$$

where ∂D is the boundary of D, and is composed of one cylinder S and two disks D_1 and D_2. Since the electric field is perpendicular to the disks D_1 and D_2, the flux through these disks vanishes, and we are left with

$$\oiint_{\partial D} \mathbf{E}d\mathbf{A} = \iint_S \mathbf{E}d\mathbf{A} = \iint_S E(r)\hat{\mathbf{r}}\cdot r\hat{\mathbf{r}}d\phi dz = rE(r)\int_0^{2\pi} d\varphi \int_{z_0}^{z_0+h} dz = 2\pi hrE(r)$$

if the cylinder occupies $z \in [z_0, z_0+h]$. Thus

$$2\pi hrE(r) = \frac{\rho h}{\epsilon_0}$$

or

$$E(r) = \frac{\rho}{2\pi r\epsilon_0}$$

so that the sought field is

$$\mathbf{E} = \frac{\rho}{2\pi\epsilon_0 r}\hat{\mathbf{r}}.$$

2.3.2 The Magnetic Field from a Stationary Current in a Straight Wire

Consider again an infinite wire along the z axis of a cylindrical coordinate system. This time, let the wire carry a constant electric current I [unit C/s] in the $\hat{\mathbf{z}}$ direction. From symmetry, the magnetic field cannot depend on z or φ. Therefore, it suffices to compute the field at $(r, \varphi, z) = (r, 0, 0)$. Also by symmetry, at any given point, the field vector cannot have a projection in the $\hat{\mathbf{z}}$ direction. However, since we have broken the symmetry by assigning a positive direction of the current along the wire, the field vector can in fact have a non-zero component in the $\hat{\boldsymbol{\varphi}}$ direction. Indeed, the right-hand rule assigns to a positive direction of the current, a positive sense of rotation around the wire.

This problem can be solved either naïvely using the Biot–Savart law, or more efficiently (again due to symmetry) using the Ampère's circuital law. We will demonstrate both approaches.

2.3.2.1 Approach 1: The Biot–Savart law

The Biot–Savart law is

$$\mathbf{B}(\mathbf{x}) = \int_\Gamma d\mathbf{B}(\mathbf{x}) = \int_\Gamma \frac{\mu_0 I d\mathbf{l} \times \hat{\mathbf{v}}(\mathbf{x}')}{4\pi r(\mathbf{x}')^2}.$$

Now Γ is a straight line, and so $d\mathbf{l}$ is constant. The integrand is a vector with the direction of $d\mathbf{l} \times \hat{\mathbf{v}}(\mathbf{x}')$. This direction is perpendicular both to the wire ($d\mathbf{l}$) and to the direction $\hat{\mathbf{v}}(\mathbf{x}')$ from the current point $\mathbf{x}'$ on the wire to the field point $\mathbf{x}$. Given a fixed field point $\mathbf{x}$, such as $(r, \varphi, z) = (r, 0, 0)$, the latter ($\hat{\mathbf{v}}(\mathbf{x}')$) lies always in the same plane. Thus, the direction of the integrand is constant, and, in addition, is given by the right-hand rule. Integrate from $z = -\infty$ to $z = \infty$. Then $d\mathbf{l} = dz\hat{\mathbf{z}}$ and so the direction of $d\mathbf{l} \times \hat{\mathbf{v}}(\mathbf{x}')$ is everywhere that of $\widehat{\boldsymbol{\varphi}}$. Consequently, $\mathbf{B}(\mathbf{x}) = B(r)\widehat{\boldsymbol{\varphi}}$, and a small contribution

$$d\mathbf{B} \coloneqq \frac{\mu_0 I}{4\pi r(\mathbf{x}')^2}\left(d\mathbf{l} \times \hat{\mathbf{v}}(\mathbf{x}')\right) = \frac{\mu_0 I dz \sin\alpha}{4\pi(z^2 + r^2)}\widehat{\boldsymbol{\varphi}}$$

where

$$\alpha \coloneqq \angle(\hat{\mathbf{z}}, \hat{\mathbf{v}}) = \arctan\frac{r}{z}$$

so that

$$\sin\alpha = \sin\arctan\frac{r}{z} = \frac{r/z}{\sqrt{1 + \left(\frac{r}{z}\right)^2}} = \frac{r}{\sqrt{z^2 + r^2}}$$

and

$$d\mathbf{B} = \frac{\mu_0 I r dz}{4\pi(z^2 + r^2)^{3/2}}\widehat{\boldsymbol{\varphi}}.$$

Therefore, the total magnetic field at $\mathbf{x}$ is

$$\mathbf{B}(r) = \int_\Gamma d\mathbf{B} = \int_{\mathbb{R}} \left(\frac{\mu_0 I r dz}{4\pi(z^2 + r^2)^{3/2}}\widehat{\boldsymbol{\varphi}}\right) = \frac{\mu_0 I r}{4\pi}\widehat{\boldsymbol{\varphi}} \int_{\mathbb{R}} \frac{dz}{(z^2 + r^2)^{3/2}} = \frac{\mu_0 I}{2\pi r}\widehat{\boldsymbol{\varphi}}.$$

Thus

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{2\pi r}\widehat{\boldsymbol{\varphi}}.}$$

2.3.2.2 Approach 2: Ampère's Circuital Law

The circuital law is superior in situations with symmetry, as is Gauss' law when it comes to electrostatics (compared with Coulomb's law), and as is Gauss' law of gravity when it comes to gravitational fields (compared with Newton's law of universal gravitation).

Indeed, since we know that the field has the form $\mathbf{B} = B(r)\widehat{\boldsymbol{\varphi}}$, we introduce a circular loop Γ of radius r around the wire, at some height. For simplicity, we take Γ to be the image of $\varphi \mapsto (r, \varphi, z) = (r, \varphi, 0)$ as $\varphi \in [0, 2\pi[$. Then the circuital law states

$$\oint_\Gamma \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$$

while

$$\oint_\Gamma \mathbf{B}\cdot d\mathbf{l} = \int_0^{2\pi} B(r)\hat{\boldsymbol{\varphi}}\cdot r d\varphi \hat{\boldsymbol{\varphi}} = B(r)r\int_0^{2\pi} d\varphi = 2\pi B(r)r.$$

Thus

$$\mu_0 I = 2\pi B(r)r$$

and so

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

and

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}}.$$

2.4 The Electromagnetic Potentials

In the electrostatic case, where there is no electric current – and therefore no magnetic field –, $\mathbf{B} = \mathbf{0}$ and so (M3) states that the electric field $\mathbf{E}$ is irrotational. Therefore, there exists a potential scalar field ϕ_{es}, defined up to an additive constant, such that

$$\mathbf{E} = -\nabla\phi_{\text{es}}.$$

This *electrostatic potential*, measured in volts, is familiar to everyone. For instance, the absolute difference $|\phi_{\text{es}}(\mathbf{r}_1) - \phi_{\text{es}}(\mathbf{r}_2)|$ in voltage between the two poles of a typical battery, located at $\mathbf{r}_1$ and $\mathbf{r}_2$, is 1.5 V. Now (↑) makes (M3) an identity, whereas (M1) yields

$$\nabla^2\phi_{\text{es}} = -\frac{\rho}{\epsilon_0}$$

which is known as *Poisson's equation*. In particular, in vacuum, $\rho \equiv 0$ and

$$\nabla^2\phi_{\text{es}} = 0$$

which is *Laplace's equation*. Given a bounded and sufficiently 'nice' region D in space, the Dirichlet problem of finding ϕ_{es} satisfying Laplace's equation in all of D and satisfying the boundary condition $\phi_{\text{es}}(\mathbf{r}) = f(\mathbf{r})$ for all $\mathbf{r} \in \partial D$, where f is a given function on the boundary ∂D, has a unique solution. A standard way of finding the solution in the *two*-dimensional case [that is, $D \subset \mathbb{R}^2 \sim \mathbb{C}$] is to employ a conformal mapping of D into some simpler region D' in which the solution is known. See (Saff & Snider, 2003) for details on this technique. [Notice that every three-dimensional problem with cylindrical symmetry is equivalent to a two-dimensional problem.]

In the non-static case, (↑) ceases to be valid. Indeed, a vector field with a non-zero rotation cannot be the gradient of *any* scalar field u, because of the vector identity $\nabla \times (\nabla u) = \mathbf{0}$. But (M4), which, as it stands, *is* the most general form of this Maxwell equation, implies that there exists a *vector potential* $\mathbf{A}$ such that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

This makes (M4) true due to the vector identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. Now, let $\mathbf{A}$ be any such vector potential, and consider the vector field

$$\boldsymbol{\Psi} \coloneqq \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}.$$

Then

$$\nabla \times \boldsymbol{\Psi} = \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = \nabla \times \mathbf{E} + \nabla \times \left(\frac{\partial \mathbf{A}}{\partial t}\right) = \nabla \times \mathbf{E} + \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \nabla \times \mathbf{E} + \frac{\partial}{\partial t}(\mathbf{B}) = \mathbf{0}$$

by the full (M3). Therefore $\boldsymbol{\Psi}$ is irrotational, and so there exists a scalar potential ϕ such that $\boldsymbol{\Psi} = -\nabla\phi$, which is well-defined up to an additive constant. Let ϕ be such a potential. Then, by the definition of $\boldsymbol{\Psi}$ we have

Proposition 1

Given an electromagnetic field $(\mathbf{E}, \mathbf{B})$, there exists a scalar field ϕ and a vector field $\mathbf{A}$ such that

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Definition 2

The scalar field ϕ and the vector field $\mathbf{A}$ of Proposition 1 are called the *scalar potential* and the *vector potential* of the electromagnetic field, respectively.

Notice that in the electrostatic case, the scalar potential ϕ reduces to the familiar electrostatic potential ϕ_{es} discussed earlier in that $\mathbf{E} = -\nabla\phi$ in this case.

The vector potential is not unique. Indeed, given a vector potential $\mathbf{A}$, consider

$$\mathbf{A}' \coloneqq \mathbf{A} + \nabla\psi$$

where ψ is any scalar field. Then

$$\nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} + \nabla\psi) = \nabla \times \mathbf{A} + \nabla \times (\nabla\psi) = \nabla \times \mathbf{A} = \mathbf{B}$$

and so $\mathbf{A}'$ is also a vector potential for $\mathbf{B}$. This freedom in choosing the vector potential is called *gauge freedom*, and any additional requirement that will fix any particular vector potential (or, at least, which restricts the class of admissible potentials) is called a *gauge choice*. What happens to the scalar potential ϕ when we replace $\mathbf{A}$ by $\mathbf{A}' = \mathbf{A} + \nabla\psi$? Note that

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi - \frac{\partial}{\partial t}(\mathbf{A}' - \nabla\psi) = -\nabla\phi - \frac{\partial \mathbf{A}'}{\partial t} + \nabla\left(\frac{\partial\psi}{\partial t}\right) = -\nabla\left(\phi - \frac{\partial\psi}{\partial t}\right) - \frac{\partial \mathbf{A}'}{\partial t}$$

and, clearly, we have to replace the old scalar potential ϕ by

$$\phi' \coloneqq \phi - \frac{\partial\psi}{\partial t}.$$

2.5 Electromagnetic Waves

Before Maxwell, the nature of light was unclear. Newton thought of light as a current of particles, but at the turn of the eighteenth century, there were also physicists that believed light to be a wave [of some sort] travelling through some medium, the so-called 'ether', in much the same way that sound travels through matter. In 1801, it is believed, the English scientist Thomas Young performed his famous double-slit experiment, demonstrating the wave-like properties of light, and, by so doing, settled the conflict in favour of the wave hypothesis. Not until the advent of quantum mechanics [Compton scattering, the photoelectric effect, etc.] did the particle nature manifest itself experimentally, to yield the 'double nature' that light is known to have today.

Here we will focus on the wave theory of light. Even if Young's experiment did support a wave theory of light, it said absolutely nothing about what kind of wave light is. After all, *any* mathematical quantity that satisfies the wave equation is called a 'wave', no matter if the quantity is the pressure in a gas (as in the case of sound), a force field, or something else.

Today we know better. Light, and, more generally, electromagnetic radiation, is a wave of the electric and magnetic fields, that is, each component of $\mathbf{E}$ and $\mathbf{B}$ satisfies the wave equation. As force fields, however, they require no medium through which to travel. Hence, there is no need for any 'ether'. In contrast, being a pressure wave, sound *does* require a medium, in which the pressure can be varied according to the art of Fourier synthesis. We will now see what led the physicists to the discovery that light is (very likely) an electromagnetic wave. [21]

First, a few definitions that extend differential operators normally applied exclusively to scalar fields to vector fields, basically by letting them act component-wise on the vector fields.

Definition 3

Let $\mathbf{A} = \left(A_x, A_y, A_z\right)$ be a *vector* field. Then we define the 'gradient' $\nabla\mathbf{A} \coloneqq (\nabla A_x \quad \nabla A_y \quad \nabla A_z)$ which is now a *matrix* whose columns are the ordinary gradients of the scalar components of $\mathbf{A}$. Furthermore, for any *matrix* A, we define the 'divergence', which is now a *vector*, by $\nabla \cdot A \coloneqq (\nabla \cdot \mathbf{A}_1, \nabla \cdot \mathbf{A}_2, \nabla \cdot \mathbf{A}_3)$ where $\mathbf{A}_i$ is the ith column of A treated as a vector. Finally, we define the 'vector Laplacian' $\nabla^2\mathbf{A} \coloneqq \nabla \cdot (\nabla\mathbf{A})$ which is now a vector.

These definitions imply

Corollary 4

Let $\mathbf{A} = \left(A_x, A_y, A_z\right)$ be a vector field. Then

$$\nabla^2\mathbf{A} = \left(\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z\right).$$

Now, consider the Maxwell equations in vacuum. They are

[21] I *hate* when upper secondary school teachers *define* light as 'a wave' without saying what it is that 'waves'. I recall one classmate of mine that said to me, "God, that's so cool – if you make sound with high enough frequency, you'll get light!" And, indeed, since both 'light' and 'sound' had been defined as 'waves', that was almost a valid conclusion.

$$\nabla \cdot \mathbf{E} = 0 \tag{MV1}$$
$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{MV2}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{MV3}$$
$$\nabla \cdot \mathbf{B} = 0. \tag{MV4}$$

Differentiation of (MV2) with respect to time yields

$$\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

and, using (MV3) to substitute for $\partial \mathbf{B}/\partial t$, we obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

It is straightforward to show the vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{VI}$$

where ∇^2 is our new vector Laplacian. In this case, the identity yields

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

But since $\nabla \cdot \mathbf{E} = 0$ we simply end up with

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2 \nabla^2 \mathbf{E} = \mathbf{0} \tag{W1}$$

which is the (vector) wave equation with speed

$$c \coloneqq \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

By Corollary 4 this means that each (scalar field) component of $\mathbf{E}$ satisfies the ordinary wave equation. Similarly, differentiation of (MV3) yields

$$\nabla \times \left(\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

where we use (MV2) to obtain

$$\nabla \times \left(\frac{1}{\mu_0 \epsilon_0} \nabla \times \mathbf{B}\right) = -\frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

The same vector identity (VI), together with $\nabla \cdot \mathbf{B} = 0$, then gives

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - c^2 \nabla^2 \mathbf{B} = \mathbf{0} \tag{W2}$$

with the *same* speed c. We have thus seen that in vacuum, Maxwell's equations imply vector wave equations for the electromagnetic field. This means that, in vacuum, we can apply all theorems on the three-dimensional wave equation to the electromagnetic field; for instance, all electromagnetic signals necessarily travel with the constant speed c. But don't be fooled – there is no equivalence between the full set of Maxwell vacuum equations (MW1-MV4) and these two wave equations. It is easy to see that this cannot be so. Indeed, the wave equations are perfectly sym-

metric in $\mathbf{E}$ and $\mathbf{B}$, but the Maxwell vacuum equations (MV2) and (MV3) are not, because $\mu_0\epsilon_0 \neq -1$.

Let us now investigate a general plane-wave solution of the Maxwell vacuum equations. We will postulate a monochromatic plane-wave $\mathbf{E}$ field and investigate the requirements on $\mathbf{E}$ and $\mathbf{B}$ imposed by the vacuum equations in this case. To this end, choose a (constant) amplitude vector $\mathbf{E}_0$, a (constant) angular frequency ω, and a (constant) wave vector $\mathbf{k}$ and set

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}_0 \sin(\omega t - \mathbf{k}\cdot\mathbf{x})$$

which is a monochromatic wave travelling in the direction of $\mathbf{k}$ with phase speed ω/k $(= \lambda\nu)$ where $\lambda = \frac{2\pi}{k}$ is the wavelength and $\nu = \frac{\omega}{2\pi}$ is the (ordinary) frequency. In this case (W1) reads

$$-\omega^2\mathbf{E}(t,\mathbf{x}) = -c^2k^2\mathbf{E}(t,\mathbf{x})$$

and is clearly satisfied iff $c = \omega/k$. *Hence, the constant* $c \overset{\text{def}}{=} 1/\sqrt{\mu_0\epsilon_0}$ *in the wave equations is indeed equal to the phase speed* ω/k *of the plane electric wave.* But what about the original vacuum equations? Now (MV1) reads

$$-(\mathbf{k}\cdot\mathbf{E}_0)\cos(\omega t - \mathbf{k}\cdot\mathbf{x}) = 0, \qquad \forall t \in \mathbb{R}, \qquad \forall \mathbf{x} \in \mathbb{R}^3$$

and, consequently, it implies

$$\mathbf{k} \perp \mathbf{E}_0.$$

The third vacuum equation, (MV3), reads

$$(\mathbf{E}_0 \times \mathbf{k})\cos(\omega t - \mathbf{k}\cdot\mathbf{x}) = -\frac{\partial \mathbf{B}}{\partial t}.$$

Therefore,

$$\mathbf{B}(t,\mathbf{x}) = \frac{1}{\omega}(\mathbf{k}\times\mathbf{E}_0)\sin(\omega t - \mathbf{k}\cdot\mathbf{x}) + \mathbf{f}(\mathbf{x})$$

for some purely spatial function $\mathbf{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$. Now, define (the constant)

$$\mathbf{B}_0 \coloneqq \frac{1}{\omega}(\mathbf{k}\times\mathbf{E}_0)$$

as[22] to obtain

$$\mathbf{B}(t,\mathbf{x}) = \mathbf{B}_0 \sin(\omega t - \mathbf{k}\cdot\mathbf{x}) + \mathbf{f}(\mathbf{x})$$

where

$$\mathbf{B}_0 \perp \mathbf{k}, \qquad \mathbf{B}_0 \perp \mathbf{E}_0.$$

The vacuum equations do not demand that $\mathbf{f}(\mathbf{x}) \equiv \mathbf{0}$ in all space; the only requirements on this term are that it is solenoidal (that is, divergence-free) and irrotational. The divergence-freeness follows immediately from (MV4) by linearity of the divergence operator and the fact that $\mathbf{B}_0 \perp \mathbf{k}$. The fact that $\mathbf{f}(\mathbf{x})$ is irrotational follows from (MV2), as we will see in a moment. These are the only requirements on $\mathbf{f}(\mathbf{x})$; any such field $\mathbf{f}(\mathbf{x})$ will work with the vacuum equations. In particular, any constant vector field $\mathbf{f}(\mathbf{x})$ is admissible.

The static field $\mathbf{f}(\mathbf{x})$ is to be interpreted as the *magnetostatic background field*. Indeed, in the case of a plane, monochromatic EM wave in an earthbound laboratory,

[22] Notice that we may write $\frac{1}{\omega}(\mathbf{k}\times\mathbf{E}_0) = \frac{k}{\omega}(\hat{\mathbf{k}}\times\mathbf{E}_0) = \frac{1}{c}(\hat{\mathbf{k}}\times\mathbf{E}_0)$ where $\hat{\mathbf{k}} \coloneqq \frac{1}{k}\mathbf{k}$.

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}_0 \sin(\omega t - \mathbf{k}\cdot\mathbf{x})$$
$$\mathbf{B}(t,\mathbf{x}) = \mathbf{B}_0 \sin(\omega t - \mathbf{k}\cdot\mathbf{x}) + \mathbf{f}(\mathbf{x})$$

where $\mathbf{f}(\mathbf{x})$ is the magnetic field of the Earth, which locally is constant to an excellent approximation. We can thus decompose the magnetic field into two parts: $\mathbf{B}_0 \sin(\omega t - \mathbf{k}\cdot\mathbf{x})$ which is the magnetic wave required to accompany the postulated electric wave $\mathbf{E}_0 \sin(\omega t - \mathbf{k}\cdot\mathbf{x})$, and a static background part $\mathbf{f}(\mathbf{x})$, which is not to be considered as a part of the EM wave. While studying the EM wave, therefore, we neglect the background field, and so we set $\mathbf{f}(\mathbf{x}) \equiv \mathbf{0}$. [Nevertheless, we will keep including a shaded term $\mathbf{f}(\mathbf{x})$ in the equations to follow just to motivate the promised result $\nabla \times \mathbf{f}(\mathbf{x}) = \mathbf{0}$.]

Now, let us move on. (MV2) gives

$$(\mathbf{B}_0 \times \mathbf{k}) \cos(\omega t - \mathbf{k}\cdot\mathbf{x}) + \nabla \times \mathbf{f}(\mathbf{x}) = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

But $\mathbf{E}$ is known, so

$$(\mathbf{B}_0 \times \mathbf{k}) \cos(\omega t - \mathbf{k}\cdot\mathbf{x}) + \nabla \times \mathbf{f}(\mathbf{x}) = \omega \mu_0 \epsilon_0 \mathbf{E}_0 \cos(\omega t - \mathbf{k}\cdot\mathbf{x}).$$

In addition,

$$\mathbf{B}_0 \stackrel{\text{def}}{=} \frac{1}{\omega}(\mathbf{k} \times \mathbf{E}_0) \Rightarrow \mathbf{B}_0 \times \mathbf{k} = \frac{k^2}{\omega}\mathbf{E}_0$$

and so

$$\frac{k^2}{\omega}\mathbf{E}_0 \cos(\omega t - \mathbf{k}\cdot\mathbf{x}) + \nabla \times \mathbf{f}(\mathbf{x}) = \omega \mu_0 \epsilon_0 \mathbf{E}_0 \cos(\omega t - \mathbf{k}\cdot\mathbf{x})$$

or

$$\left(\frac{\omega^2}{k^2}\mu_0 \epsilon_0 - 1\right)\mathbf{E}_0 \cos(\omega t - \mathbf{k}\cdot\mathbf{x}) = \nabla \times \mathbf{f}(\mathbf{x}).$$

Recalling that $\mathbf{f}(\mathbf{x}) \equiv \mathbf{0}$, we thus regain the result

$$\frac{\omega^2}{k^2}\mu_0 \epsilon_0 = c^2 \mu_0 \epsilon_0 = 1.$$

On the other hand, if we consider the static field $\mathbf{f}(\mathbf{x})$ [that is, stop neglecting it!], and recall that the above relation already has been established, then it follows that $\mathbf{f}(\mathbf{x})$ is irrotational, as promised. As indicated above, any constant field $\mathbf{f}(\mathbf{x})$, such as the locally constant earthly magnetic field, is solenoidal and irrotational, and thus provides a concrete illustration of the static background field.

Conversely, if we had begun by postulating a monochromatic, plane-wave, *magnetic* field, then we would have obtained a solenoidal and irrotational *electro*static background field term, in addition to the electric wave that needs to accompany the magnetic wave, that is

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}_0 \sin(\omega t - \mathbf{k}\cdot\mathbf{x}) + \mathbf{f}(\mathbf{x})$$
$$\mathbf{B}(t,\mathbf{x}) = \mathbf{B}_0 \sin(\omega t - \mathbf{k}\cdot\mathbf{x}).$$

Now, a vector field $\mathbf{f}(\mathbf{x})$ that is both divergence-free and irrotational is called a 'Laplacian' vector field. Because it is irrotational, it is a potential field, and thus the gradient of a scalar potential $\phi(\mathbf{x})$. But since $\nabla \cdot \mathbf{f}(\mathbf{x}) = \nabla \cdot \big(\nabla\phi(\mathbf{x})\big) = \nabla^2\phi(\mathbf{x}) = 0$, the potential is harmonic. That is, the static

background field has to be the gradient of a harmonic function. But this is exactly what we would expect, because *the electrostatic potential* is harmonic (in vacuum)!

In any case (postulating a plane-wave electric field *or* postulating a plane-wave magnetic field), the background term needs to be *static*; it may not vary in time. This is because of the interactions between the electric and magnetic fields. Indeed, a time-varying magnetic field would affect the electric field (which we have already fixed), and, conversely, a time-varying electric field would affect the magnetic field.

To summarize, we have found that:

- Maxwell's equations admit plane electromagnetic (EM) waves.
- A plane **E** field wave must be accompanied with an equally plane magnetic **B** field wave (and vice-versa). These waves are oscillating *in phase*.
- A plane EM wave must be transversal, i.e. $\mathbf{E}_0 \perp \mathbf{k}$ and $\mathbf{B}_0 \perp \mathbf{k}$. That is, the **E** and **B** fields are always perpendicular to the wave vector.
- The electric field (wave) is perpendicular to the magnetic field (wave), i.e. $\mathbf{B}_0 \perp \mathbf{E}_0$.
- An electromagnetic wave has the constant phase speed $\frac{1}{\sqrt{\mu_0\epsilon_0}}$. From now on, we will denote this *speed of light* by the symbol c_0. This is equal to the constant in the wave equations (W1-W2) for the electromagnetic field.
- $E = c_0 B$. [This follows from $\mathbf{B}_0 \overset{\text{def}}{=} \frac{1}{c_0}\left(\hat{\mathbf{k}} \times \mathbf{E}_0\right)$.]

Maxwell's theory thus gives an explicit formula for the speed of an electromagnetic wave in terms of fundamental physical quantities. Around 1862, one knew the speed of light to a reasonable accuracy, and one noticed that it was rather close to the (also known) numerical value of $1/\sqrt{\mu_0\epsilon_0}$, which, combined with the newly published Maxwell theory, strongly suggests that light is in fact an EM wave, and during the years to come, all doubt was removed: *Light is an electromagnetic wave!*

2.5.1 The 'Speed' of Light

Let's resolve a minor difficulty that many texts simply pretend doesn't exist [many texts also 'forget' to motivate why one should set the constant of integration $\mathbf{f}(\mathbf{x}) \equiv \mathbf{0}$ above]: As we have seen, $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ is the *phase* speed of light. However, it wasn't the *phase* speed of light that was known (approximately) in 1862; instead, it was the *speed of propagation* of an electromagnetic signal. This is the constant c in the wave equations (W1)-(W2), but, again as we have seen, this is equal to the phase speed c_0.

Indeed, the wave equations (W1)-(W2) are satisfied in *all* of (empty) space, and so the electromagnetic field has to satisfy them both 'inside' a beam of light, and in the surroundings. But it is a fundamental theorem on the three-dimensional wave equation that any signal travels at the speed c, no more and no less. More precisely, we have

Theorem 6

Let $u(\mathbf{x}, t)$ be a scalar field in space-time satisfying the wave equation $u_{tt} - c^2\nabla^2 u = 0$ everywhere and the initial conditions $u(\mathbf{x}, 0) = \phi(\mathbf{x})$ and $u_t(\mathbf{x}, 0) = \psi(\mathbf{x})$. Then

$$u(\mathbf{x},t)=\frac{1}{4\pi c^2 t}\iint_S \psi(\mathbf{x})dS+\frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint_S \phi(\mathbf{x})dS\right]$$

where $S=\{\mathbf{x}'\in\mathbb{R}^3\colon|\mathbf{x}-\mathbf{x}'|=ct\}$.

The proof of this theorem can be found in (Walter, 2008, pp. 234-238). Notice that the solution at any given point $(\mathbf{x},t)$ only depends on the value of the initial conditions a distance ct away from the point. That is, information about the initial conditions travels at the speed c. Since the EM field at any particular moment of time can be used as the initial conditions in the problem of finding the field at later times, it follows, for instance, that a beam of light has to travel with precisely the phase speed c_0 of the electromagnetic wave.

2.6 Inconsistencies with Classical Mechanics

2.6.1 The Speed of Light

Let A be an observer standing at rest on the ground of the Earth, and let B be a passenger in a space ship travelling with velocity $\mathbf{v}$ with respect to A and the ground. Imagine that B fires a cannon ball in his direction of motion. Relative to B and the space ship, the velocity of the cannon ball is $\mathbf{u}$. According to A, of course, the velocity of the cannon ball is $\mathbf{v}+\mathbf{u}$, and the speed $v+u>u$. This is self-evident from a Newtonian point of view.

Now, let B fire a laser gun instead. According to B, the laser beam travels with speed c_0. Therefore, classically, one would expect A to observe the beam travelling with a slightly higher speed, namely $v+c_0$. Of course, since $c_0 \gg v$, we have $v+c_0 \approx c_0$ and the difference is barely detectable, if at all – but it is there. Nevertheless, A knows electrodynamics, and Maxwell's equations are very clear: relative to A, the beam *has* to travel with speed c_0, no less and no more.

The reader might think this is not a fundamental problem simply because light is profoundly different from matter, e.g. from a cannon ball. Nevertheless, it is! Later on, when introducing the special theory of relativity, we will see exactly how fundamental a problem this is; the ramifications are overwhelming. Of course, since c_0 is such a great number, the discrepancy between the classical expectations and the real case is most often invisible to us. But at high speeds, when $v \ll c_0$ ceases to be valid, so does $v+c_0 \approx c_0$, and all the strange effects of special relativity reveal themselves.

On a historical note, the difficulty associated with the constant speed of light as predicted by Maxwell's equations was known from the beginning. The suggested 'solution' was that there existed a *preferred* frame of reference, and that Maxwell's equations were *only* exactly valid in this frame. This preferred frame was supposed to be the frame in which the 'ether' was at rest. In other words, c_0 was the speed of light relative to the 'ether', and an observer not at rest relative to this mysterious 'ether' would measure a *different* speed of light, in accordance with the classical 'common sense'. Since the Earth moved through the 'ether', it would have been possible to detect the slight change of the speed of light during the year. The most famous experiment attempting to detect this variation was the Michelson-Morley experiment of 1887 (Tipler & Llewellyn, 2008). No variation was detected, however. Today the 'ether' hypothesis is (almost) completely abandoned.

2.6.2 The Galilean Transformation

The abnormal constancy of the speed of light is perhaps the most intuitive case against the combination Maxwell + Newtonian Mechanics. On a less intuitive note, a severe theoretical problem with the same combination is that Maxwell's equations are not invariant under Galilean transformations. That is, if Maxwell's equations are valid in a physical frame $\wp\mathcal{F}1$, then they are *not* necessarily valid in any other physical frame $\wp\mathcal{F}2$! This, of course, is a major problem. Indeed, if Maxwell's equations are valid in at most one physical reference frame, then, the odds that they are valid in your laboratory frame are practically zero. We now wish to prove this. The simplest way is perhaps to employ a proof by contradiction.

Let $\mathcal{F}1 \in \wp\mathcal{F}1$ be a system at rest relative to the laboratory (the unprimed system). Let there be a stationary wire with constant charge density $\rho > 0$ [C/m] along the y axis carrying no electric current. Place a test charge $q > 0$ a distance $d > 0$ below the wire, as in Figure 27. Let $\mathcal{F}2 \in \wp\mathcal{F}2$ be a system with the same basis vectors but moving with velocity $\mathbf{v} = v\hat{\mathbf{y}}_1$ relative to $\mathcal{F}1$.

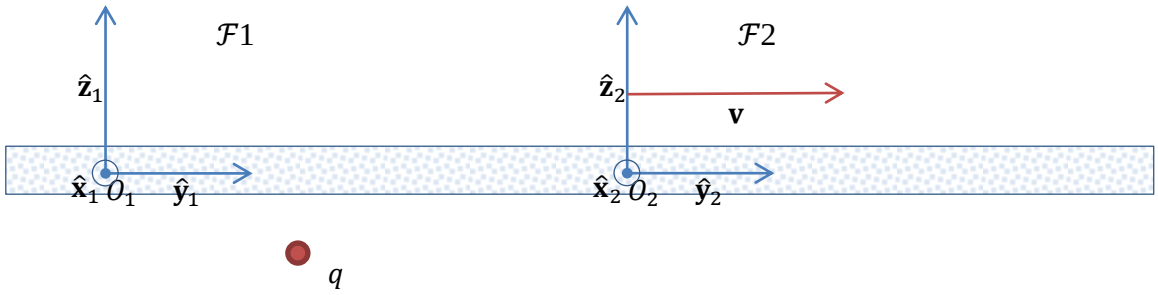


Figure 27. A charge below a wire.

Assume that the electric and magnetic fields are $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{E}', \mathbf{B}')$ in $\mathcal{F}1$ and $\mathcal{F}2$, respectively. Further, assume that Maxwell's equations are valid in both frames. In $\mathcal{F}1$, (M1) yields

$$\mathbf{E} = -\frac{\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_1$$

at the test particle. (M2) and (M4) state that the magnetic field is a Laplacian vector field. But (M3) requires more than that: since the electric field is irrotational, the magnetic field has to be constant in time. Let us assume that there is no constant background field, that is, set $\mathbf{B} = \mathbf{0}$ identically. Therefore, all things considered, the charge will experience the force

$$\mathbf{F} = q\mathbf{E}$$

and it will obtain an (instantaneous) acceleration

$$\mathbf{a} = \frac{1}{m}\mathbf{F} = \frac{q}{m}\mathbf{E} = -\frac{q\rho}{2\pi\epsilon_0 md}\hat{\mathbf{z}}_1.$$

Now, according to an observer in $\wp\mathcal{F}2$, the wire carries a constant current $\mathbf{J} = -\rho\mathbf{v}$ which corresponds to a scalar current of $I = |\boldsymbol{J}| = \rho v$. Therefore, in addition to the electrostatic field

$$\mathbf{E}' = -\frac{\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_2$$

given by (M1), there is also a static (because of (M3)) non-vanishing magnetic field, given by (M2), namely

$$\mathbf{B}' = \frac{\mu_0 I}{2\pi d}\hat{\mathbf{x}}_2.$$

In addition, since the test charge is moving with velocity $-\mathbf{v}$ with respect to $\wp\mathcal{F}2$, it will be affected by this magnetic field, and so the charge will experience the total Lorentz force

$$\mathbf{F}' = q\mathbf{E}' + q(-\mathbf{v}) \times \mathbf{B}'$$

and thus the (instantaneous) acceleration

$$\mathbf{a}' = \frac{1}{m}\mathbf{F}' = \frac{q}{m}\mathbf{E}' - \frac{q}{m}\mathbf{v} \times \mathbf{B}' = -\frac{\rho q}{2\pi m\epsilon_0 d}\hat{\mathbf{z}}_2 - \frac{q}{m}v\hat{\mathbf{y}}_2 \times \frac{\mu_0 I}{2\pi d}\hat{\mathbf{x}}_2 = -\frac{\rho q}{2\pi m\epsilon_0 d}\hat{\mathbf{z}}_2 + \frac{qv\mu_0 I}{2\pi md}\hat{\mathbf{z}}_2 =$$
$$= -\frac{q\rho}{2\pi\epsilon_0 md}(1 - v^2\mu_0\epsilon_0)\hat{\mathbf{z}}_2 = \mathbf{a} + \frac{q\rho v^2\mu_0}{2\pi md}\hat{\mathbf{z}}_2$$

But since $\mathcal{F}1$ and $\mathcal{F}2$ are both inertial frames, $\mathbf{a}' = \mathbf{a}$ as geometric vectors (and in this case the components have to agree, too). Therefore, $q\rho v^2\mu_0 = 0$. But since $q > 0$, $\rho > 0$, and $v > 0$ [and, of course, $\mu_0 > 0$], this is a contradiction.

What have we shown? We have shown that, if Maxwell's equations are valid in one inertial frame, they are *not* valid in 'any' [at least not every] other such frame. But since common sense requires all inertial frames to be equivalent (since it is highly dubious to think that the frame of one's personal laboratory is a God-given frame of magic), we have thus shown that *Maxwell's equations cannot be the truth [unless there is something very strange going on]*. Einstein, however, was brave enough to question the 'common sense'. He based his special theory of relativity on the assumption that Maxwell's equations *are* in fact valid, and that it is the common-sense Galilean transformation that is at fault. Indeed, as we shall see below, when one considers relativistic effects such as time dilation and length contraction, Maxwell's equations will be equally valid in any inertial frame.

> **Exercise:** Motivate why the discrepancy illustrated above is very small in a typical every-day experiment.

3 Special Relativity

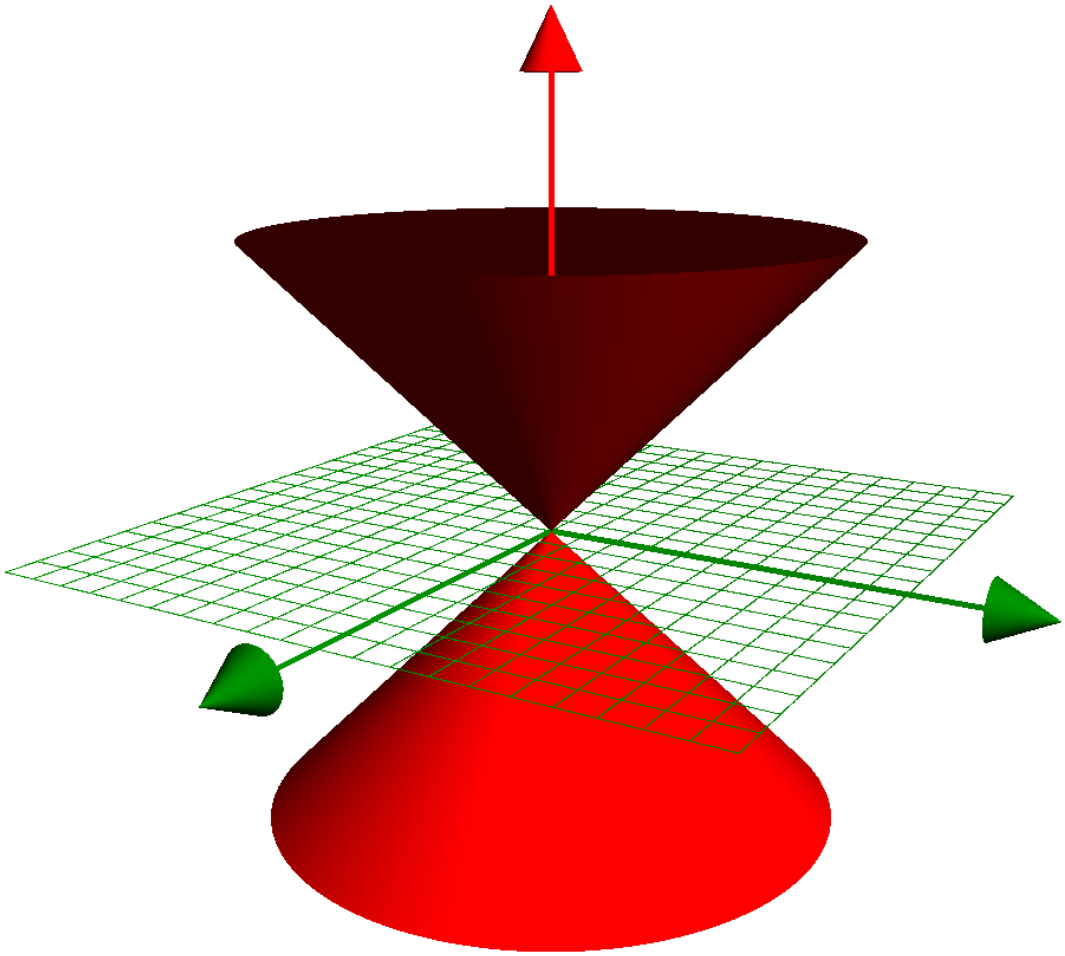


Figure 28. The 'Light Cone', which we will introduce in this chapter, is perhaps the most important visual aid in understanding the geometry of flat spacetime.

3.1 Time Dilation

Contemporary texts on special relativity tend to base the introduction on the so-called 'k-calculus'. Examples include (d'Inverno, 1992) and (Ludvigsen, 1999). This is probably a clear and stringent approach. The current author, however, feels that this approach obscures the physical insights that underpin special relativity, and will therefore present the subject in a more classical manner. For the remainder of this chapter, a physical reference frame is assumed to be inertial, unless the (possibility of the) opposite is stated explicitly.

Let $\wp\mathcal{F}2$ be a high-speed aeroplane in which we use a mathematical frame $\mathcal{F}2$. Let there be a combined light emitter/detector E/D positioned at the floor, and a mirror M on the ceiling right above E/D. Let the height of the cabin, or, more precisely, the distance between E/D and M, be $h > 0$. Suppose that the emitter emits a light beam at time $t' = 0$. The beam will reflect in M and be detected by the detector at a later time $t' = T_A$, that is, T_A is the length of time that pass between the emission and the detection of the beam, as measured inside the aeroplane ['A' as in 'aeroplane']. Since light travels with the constant speed c_0 and the total distance travelled is $2h$, this duration is $T_A = 2h/c_0$.

On the other hand, according to an observer stationary with respect to the ground $\wp\mathcal{F}1$ [mathematical system $\mathcal{F}1$], the aeroplane travels (horizontally) with speed $v > 0$. Again, let the beam be emitted at time $t = 0$, and let the detection happen at time $t = T_G$, according to the stationary observer ['G' as in 'ground']. During this time, the emitter/detector E/D has moved a distance $D = vT_G$. Therefore, as perceived by this observer, the beam of light has travelled along two of the sides of an isosceles triangle with height h and base D, as in FIG.

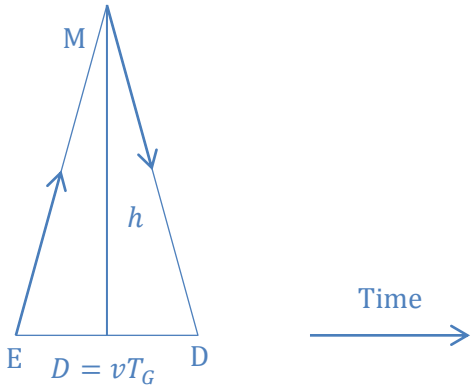


Figure 29. A beam of light in a moving aeroplane.

Classically, we would say that the speed of the light beam has increased; indeed, the velocity still would have a vertical component c_0, but also a new, horizontal, component v, so that the speed would have increased to $\sqrt{c_0^2 + v^2} > c_0$, even though the difference would be very small even for a high-speed aeroplane. But according to Maxwell's equations, the speed of light is c_0. Period. Therefore, the stationary observer will find that the beam has travelled a distance[23] $d = 2\sqrt{\left(\frac{vT_G}{2}\right)^2 + h^2}$ with constant speed c_0. Consequently, the elapsed time satisfies

[23] We assume that both observers agree on the distance between the emitter and detector being equal to h. Notice that the displacement vector between the emitter and detector is *perpendicular* to the velocity of $\wp\mathcal{F}2$ relative to $\wp\mathcal{F}1$. [Later on we will see that distances with a non-zero component in this later direction are dependent upon the observer due to 'length contraction'.] All experiments agree on the fact there

$$T_G = \frac{d}{c_0} = \frac{2}{c_0}\sqrt{\left(\frac{vT_G}{2}\right)^2 + h^2}.$$

Notice that an observer in the aeroplane $\wp\mathcal{F}2$ and an observer stationary with respect to the ground $\wp\mathcal{F}1$ disagree on how much time that elapsed between the emission and the detection of the light beam [if you don't see this, notice that T_G clearly depends on v, while T_A does not]! Clearly, this classical 'absurdity' steams from the 'absurd' constancy of the speed of light between different inertial frames of reference.

But if we are to believe in the validity of the Maxwell equations, and, in addition, not be tempered to restrict the validity of the equations to any 'preferred' frame of reference, then, time is not an absolute thing.

This forms the basis of special relativity. In fact, Einstein *postulated*

Einstein's Postulates of Special Relativity[24]

1) Every *inertial* frame of reference is equivalent to any other inertial frame of reference. Any *pair* of inertial reference frames is characterized by their relative speed.

2) Contrary to classical expectations, in *any* inertial reference frame, the speed of light is c_0.

As we have seen, these postulates imply that the 'rate of time' is dependent upon the observer. We will now solve (↑) for T_G. Some simple algebra yields

$$T_G = \frac{2}{c_0}\sqrt{\left(\frac{vT_G}{2}\right)^2 + h^2} \Rightarrow T_G^2 = \frac{4}{c_0^2}\left(\frac{v^2T_G^2}{4} + h^2\right) \Leftrightarrow T_G^2\left(1 - \frac{v^2}{c_0^2}\right) = \frac{4h^2}{c_0^2}.$$

Since $T_G^2 > 0$ and $4h^2/c_0^2 > 0$, we must have $1 - v^2/c_0^2 > 0$. Therefore, Einstein's postulates inevitable *requires*

$$v < c_0;$$

in other words, the speed of an aeroplane, as measured from the ground, is forbidden to equal or exceed the speed of light! Recalling that $T_A = 2h/c_0$ we end up with

$$T_G = \gamma(v)T_A$$

where the *Lorentz factor*

$$\gamma(v) \coloneqq \frac{1}{\sqrt{1 - \frac{v^2}{c_0^2}}} \geq 1$$

essentially forms the quantitative basis of special relativity. We remark that $\gamma(v)$ is very close to unity for all every-day speeds[25], and so Newtonian mechanics is an excellent *approximation* in these cases. However, for high-speed particles, the speed of which might come close to the speed

is no length contraction perpendicular to the direction of motion. In addition, there is no theoretical 'need' for any, either.

[24] The sentence starting "Any pair of inertial frames…" is the invention of the current author, who feels that this is an implicit assumption that is worth making explicit.

[25] You have to sit down and compute $\gamma(v)$ for some different $v \in [0, c_0[$ to get a feeling for it.

of light, the relativistic effects become quite apparent. In fact, $\gamma(v) \to \infty$ as $v \to c_0$, subtly reminding us that there cannot be any speed greater than the speed of light.

The physical phenomenon that the 'rate of time' is not absolute is called *time dilation*, and – quite apparently – is characterized by the fact that moving clocks tick more slowly than stationary clocks. Very concretely: Suppose that you have two atomic clocks that you synchronize, and that you keep one of them stationary at the ground while the other is placed in a high-speed orbit around the Earth for a few laps.[26] Then you bring it back to the stationary clock, and you compare their readings. You will find that the clock that has been in orbit will have recorded fewer ticks than the clock that has remained stationary.[27] This has been verified by experiments using atomic clocks.[28]

We wish to make the notion of time dilation more precise. To this end, we consider two frames of references, $\mathcal{F}1$ and $\mathcal{F}2$. Let $\mathcal{F}1$ be an airport on Earth, while $\mathcal{F}2$ is a spaceship. Assume that, at the origin of each, there is an observer carrying a clock. Assume that the spatial origins of $\mathcal{F}1$ and $\mathcal{F}2$ coincide at some time, that is, the spaceship is standing by on the airport at this time. Assume also, that the spaceship is set on a journey to interstellar space, and then returns to Earth after some time. At this time, the spatial origins of $\mathcal{F}1$ and $\mathcal{F}2$ once again coincide. The observer attached to the ground ($\mathcal{F}1$) will say that the ship was away for some time T_G, while the observer onboard the ship will say he was away for some time T_A.

The clock onboard the ship is measuring the *proper time* of the ship ($\mathcal{F}2$), while the clock on the airport is measuring the proper time of the airport ($\mathcal{F}1$), where we have used

Definition NN

The *proper time* of an observer is the time displayed by a clock always carried with the observer. In particular, this means that the clock is always at rest relative to the observer, and that the clock is always at the same place as the observer.

[26] This seemingly simple thought experiment is deceptively complicated, for neither the reference frame of the ground nor the frame of the orbiter is an inertial frame. Let us assume that this is not a problem. Then, since the Earth is rotating around its axis, anticlockwise as seen from above the North Pole, a point on the Earth's equator is moving with speed $u = R\omega$ to the east, where $R > 0$ is the equatorial radius of the Earth, and $\omega > 0$ is the angular velocity. Thus, a point on the equator will experience time dilation relative to the inertial ambient space because of the spinning of the Earth itself! In addition, if an equatorial aeroplane is travelling eastwards with speed $v > 0$ relative to the ground, then it will suffer from an even stronger time dilation than a fixed point on the equator. Indeed, relative to the inertial ambient space, its speed is $R\omega + v$. On the other hand, if the aeroplane is travelling with speed $v \in]0, 2R\omega[$ to the *west*, it will compensate for the Earth's rotation, and will experience a smaller effect of time dilation as compared to a fixed point on the equator, everything as perceived by the inertial observer above the North Pole.

[27] In this sentence, where we state that the aeroplane will suffer from a greater time dilation than a fixed point on the ground, we assume that it is either travelling to the east, or is travelling to the west with a speed $v > 2R\omega$. Simply put, we say that the aeroplane is travelling with high speed as compared to the rotation of the Earth, which we want to neglect. At any rate, isn't there a problem that both the clock on the fixed point on the equator and the clock inside the aeroplane are accelerating? Maybe. But experimental verification tells us that this acceleration does not invalidate the effects of time dilation. And, clearly, you can device thought experiments that involve no circular motion at all, if you wish.

[28] In October 1971, four caesium beam atomic clocks were flown around the world in commercial jet aircrafts, both eastwards and westwards. They recorded time dilation effects, opposite in the two directions, to a high degree compatible with the predictions of special (and general) relativity. The results were presented in (Hafele & Keating, 1972) and (Hafele & Keating, 1972).

The *proper time* of a mathematical frame of reference is the proper time of an observer always sitting at the spatial origin of the frame. In particular, this means that the clock, carried by the observer, is always at rest relative to the frame, and has always the same spatial coordinates with respect to it.

Let us now formulate a precise result on time dilation:

Proposition 7

Assume Einstein's postulates. Consider two *inertial* physical frames $\wp\mathcal{F}1$ and $\wp\mathcal{F}2$ with relative speed v. Let T' be the duration, as measured in $\wp\mathcal{F}2$, of a process, the starting and ending events of which occur at the same spatial coordinates in any chosen mathematical frame $\mathcal{F}2 \in \wp\mathcal{F}2$. Then

$$T = \gamma(v)T'$$

is the duration of the process as measured in any system $\mathcal{F}1 \in \wp\mathcal{F}1$.

This is what we proved above, but the hypothesis can be slightly weakened:

Proposition 8

Let $\wp\mathcal{F}1$ and $\wp\mathcal{F}2$ be any inertial physical frames with relative speed v. Choose mathematical frames $\mathcal{F}1 \in \wp\mathcal{F}1$ and $\mathcal{F}2 \in \wp\mathcal{F}2$ in standard configuration [that is, $\mathbf{v} = v\hat{\mathbf{x}}$]. Let $\mathcal{E}1$ and $\mathcal{E}2$ be two events with the same x coordinate in $\mathcal{F}2$. Let T and T' be the duration of time passing between the events as measured in $\mathcal{F}1$ and $\mathcal{F}2$, respectively. Then $T = \gamma(v)T'$.

Proof

Let $\mathcal{F}1 \in \wp\mathcal{F}1$ (the ground) and $\mathcal{F}2 \in \wp\mathcal{F}2$ (an aeroplane) be in standard configuration with relative speed v. Let there be a photon emitter at some point inside the aeroplane, and a detector a distance $h > 0$ from the emitter, but at the same x coordinate. Let $\mathcal{E}1$ and $\mathcal{E}2$ be the emission and detection events, respectively. As seen inside the aeroplane, the photon travels a distance h with speed c_0; thus, the duration between $\mathcal{E}1$ and $\mathcal{E}2$ is $T' = h/c_0$. As seen from the ground, the distance is $\sqrt{h^2 + (vT)^2}$, because, being a displacement perpendicular to the relative velocity, h is the same; thus, the duration between $\mathcal{E}1$ and $\mathcal{E}2$ is $T = c_0^{-1}\sqrt{h^2 + (vT)^2}$, which is solved to yield $T = h/\sqrt{c_0^2 - v^2}$. Therefore,

$$\frac{T}{T'} = \frac{h}{\sqrt{c_0^2 - v^2}} \cdot \frac{c_0}{h} = \gamma(v)$$

as promised. ■

We cannot, however, remove the hypothesis on the events entirely:

Observation 9

There exists a pair $(\wp\mathcal{F}1, \wp\mathcal{F}2)$ of inertial physical frames with relative speed v and a pair $(\mathcal{E}1, \mathcal{E}2)$ of events such that the duration T between the events as measured in $\wp\mathcal{F}1$ and the duration T' between the events as measured in $\wp\mathcal{F}2$ satisfy

$$T \neq \gamma(v)T'.$$

Proof

Choose two mathematical frames $\mathcal{F}1 \in \wp\mathcal{F}1$ and $\mathcal{F}2 \in \wp\mathcal{F}2$ in standard configuration. Emit a photon at time $t' = t = 0$ at the common origin in the $\hat{\mathbf{x}}'$ direction. This is the initial event $\mathcal{E}1$. Let there be a detector at coordinates $(\tilde{x}', 0, 0)$ inside the ship (relative to $\mathcal{F}2$). Let the photon be detected at time $t' = \tilde{t}'$ (relative to $\mathcal{F}2$). Thus, the detection event has coordinates $(\tilde{t}', \tilde{x}', 0, 0)$ in $\mathcal{F}2$; this is the final event $\mathcal{E}2$. According to an $\mathcal{F}1$-bound observer, at the time $t = \tilde{t}$ of detection, the detector is found at coordinate $\tilde{x}$. Thus, the coordinates of $\mathcal{E}2$ are $(\tilde{t}, \tilde{x}, 0, 0)$ relative to $\mathcal{F}1$.

Notice that the lifetime of the photon, or the duration of the process with initial and final events $\mathcal{E}1$ and $\mathcal{E}2$, is $\tilde{t}'$ relative to $\wp\mathcal{F}2$ and $\tilde{t}$ relative to $\wp\mathcal{F}1$. Notice also that Proposition 8 does not apply since the x coordinates of $\mathcal{E}1$ and $\mathcal{E}2$ differ in $\wp\mathcal{F}2$. Assume that $\tilde{t} = \gamma(v)\tilde{t}'$ anyway. Since the speed of light is c_0 in both systems,

$$\frac{\tilde{x}}{\tilde{t}} = \frac{\tilde{x}'}{\tilde{t}'} = c_0$$

which, since $\tilde{t} = \gamma(v)\tilde{t}'$, implies

$$\tilde{x} = \gamma(v)\tilde{x}'$$

independent of t, which is absurd [think about the case $\gamma(v) \sim 1$]. Therefore, $\tilde{t} \neq \gamma(v)\tilde{t}'$. ■

We end this subsection with a simple observation.

Observation NN

Inside any given, inertial, physical frame $\wp\mathcal{F}$ of reference, time is absolute. To see this, imagine that, at every point in space, there is an observer, carrying a clock, at rest relative to $\wp\mathcal{F}$. These observers can, at any time, synchronize their clocks in a very simple way. First, they need to decide on a 'team leader', and then every team member will need to determine his distance d from the team leader. This can be done by means of a ruler, in principle. Now, to synchronize the clocks, the team leader resets his clock at the same time as he emits a flash of light in every direction. Each team member will see this flash at some later time, and when he does, he resets his clock. To compensate for the light travel time, he then subtracts an amount of d/c_0 seconds from his clock. Now all clocks are synchronized.

Consider now any two events, $\mathcal{E}1$ and $\mathcal{E}2$. At the spatial position of $\mathcal{E}1$, fortunately, there sits a team member, who makes a note about the current time of the event, according to him. He can then 'broadcast' this timestamp, that is, emit an electromagnetic signal encoding it, which will be seen by all other observers. The same applies to $\mathcal{E}2$.

This way, *any* observer in $\wp\mathcal{F}$ can measure the duration t between any pair of events. In addition, it should be clear that, if some other observer in $\wp\mathcal{F}$ does the same and obtains the value t', then $t = t'$.

3.1.1 Non-Inertial Frames

Of course, a general reference frame is not inertial. Still, Definition NN applies: even if a clock *is* accelerated, it will tick, so the term 'proper time' is well-defined. We will assume that no *new* strange physical effect reveal itself when a clock is accelerated. Using this assumption, sometimes called the 'clock hypothesis', we may approximate a journey with a smooth but non-constant velocity function by a journey with a piecewise constant velocity function.

To make this precise, let $\mathcal{F}1$ be an inertial frame, and let $\mathcal{F}2$ be the frame of a spaceship, not necessarily accelerating. We will use t to denote proper (or 'coordinate') time in $\mathcal{F}1$, and τ to denote proper time of $\mathcal{F}2$. We wish to find the relation between τ and t at any time inside some arbitrary interval $[t_A, t_B]$ of $\mathcal{F}1$ time. To this end, we use the clock hypothesis and partition the interval $[t_A, t_B]$ on the $\mathcal{F}1$ coordinate time axis into N small parts $I_1 \coloneqq [t_0, t_1]$, $I_2 \coloneqq [t_1, t_2]$, ..., $I_N \coloneqq [t_{N-1}, t_N]$ where $0 = t_A < t_1 < \cdots < t_{N-1} < t_N = t_B$. Introduce $\Delta \coloneqq \max_{1 \le i \le N} \mu(I_i)$ where $\mu([a,b]) \coloneqq b - a$ as a measure of the fineness of the partition.

Let $u(t)$ denote the relative speed between the frames at time t relative to $\mathcal{F}1$. Consider some interval I_i of time, $1 \le i \le N$. In this interval, the relative speed (and velocity) is, assuming Δ is small enough, essentially constant, namely, $u(t_i)$. Let $\Delta\tau_i$ be the increase in the proper time of $\mathcal{F}2$ in this interval of $\mathcal{F}1$ time (so that, classically, $\Delta\tau_i = \mu(I_i)$). Then, using Proposition 7, which holds inside this interval since (1) $\mathcal{F}2$ has constant relative velocity relative to an inertial frame and thus is inertial itself in this interval of time, and since (2) the clock at the origin of $\mathcal{F}2$ has constant spatial coordinates relative to $\mathcal{F}2$,

$$\Delta\tau_i \approx \frac{1}{\gamma\big(u(t_i)\big)}\mu(I_i)$$

and, therefore,

$$\tau = \sum_{i=1}^{N} \Delta\tau_i \approx \sum_{i=1}^{N} \frac{1}{\gamma\big(u(t_i)\big)}\mu(I_i).$$

But the error should (this is the 'clock hypothesis') tend to zero as $\Delta \to 0$. Thus,

$$\tau = \int_{t_A}^{t_B} \frac{1}{\gamma\big(u(t)\big)}\,dt$$

and we have proved

Proposition NN

Let $\mathcal{F}1$ be an inertial frame, and let $\mathcal{F}2$ be any frame (inertial or not). Let the speed of $\mathcal{F}2$ relative to $\mathcal{F}1$ be $u(t)$ at proper time t relative to $\mathcal{F}1$. Then an amount

$$\tau = \int_{t_A}^{t_B} \frac{1}{\gamma\big(u(t)\big)}\,dt$$

of proper time is measured in $\mathcal{F}2$ between times t_A and t_B relative to $\mathcal{F}1$.

Corollary NN

Let $\mathcal{F}1$ be an inertial frame, and let $\mathcal{E}1$ and $\mathcal{E}2$ be two events both taking place at the spatial origin of $\mathcal{F}1$. Let $\mathcal{F}2$ be *any* frame of reference, only restricted by the requirement that $\mathcal{E}2$ and $\mathcal{E}2$ both should take place at the origin of $\mathcal{F}2$, too. Let t and τ be the proper times spent between the two events, relative to $\mathcal{F}1$ and $\mathcal{F}2$, respectively. Then

$$t \ge \tau.$$

Proof

Let $\mathcal{F}1$ and $\mathcal{F}2$ be as in the corollary. Let the time coordinates of $\mathcal{E}1$ and $\mathcal{E}2$ be t_1 and t_2, respectively, relative to $\mathcal{F}1$, where $t_2 \geq t_1$. Thus, $\mathcal{F}1$ finds the proper time between the events to equal $t_2 - t_1$. But, using Proposition NN, an amount of proper time

$$\tau = \int_{t_1}^{t_2} \frac{1}{\gamma\big(u(t)\big)} dt \leq t_1 - t_2$$

will pass between $\mathcal{E}1$ and $\mathcal{E}2$ relative to $\mathcal{F}2$. ■

3.2 Length Contraction

Suppose that a spaceship makes a journey along the straight line from the Earth to Vega. For simplicity, we assume that the ship travels with constant speed v relative the galaxy [or some other suitable 'background']. An observer at rest relative to the galaxy will measure the time T_G of the entire journey, which he observes has length $d_G = vT_G$, but the captain of the ship will experience a smaller duration of the flight, namely, $T_A = \frac{1}{\gamma(v)} T_G < T_G$, according to Proposition 7. [The spatial coordinates of the captain's chair are the same at both the start and the end of the journey relative to the ship.] In addition, since the captain and the earthbound observer agree on their relative velocity being v, the captain must conclude that he has only travelled a distance

$$d_A = vT_A = v\frac{1}{\gamma(v)}T_G = v\frac{1}{\gamma(v)}\frac{d_G}{v} = \frac{1}{\gamma(v)}d_G < d_G.$$

Hence, the captain observes that the galaxy shrinks as he zooms across it! This is *length contraction*. The length of an object thus depends upon the observer. The greatest length of an object is called its *proper length* and is measured in the *rest frame* of the object.

There is a rather beautiful combined illustration and physical verification of the phenomena of time dilation and length contraction, namely the journey of muons through the Earth's atmosphere. Muons are formed at high altitude by particle reactions initiated by cosmic rays.[29] However, the altitudes we are talking about are in the order of magnitude of several thousand meters above sea level, and the mean lifetime of a muon is only about 2 μs. Hence, a typical muon with a speed of $v = 0.998c_0$ (Tipler & Llewellyn, 2008) would only survive for some 600 meters. Therefore, classically, a muon reaching the ground should be a very rare event, something worth celebrating by a huge party, really.[30]

However, the particle will experience very strong relativistic effects due to its high speed. [Formally, let $\wp\mathcal{F}1$ be the ground, and $\wp\mathcal{F}2$ the rest frame of a muon, in which the muon is at rest; in particular, the muon has constant spatial coordinates in any frame $\mathcal{F}2 \in \wp\mathcal{F}2$.] The Lorentz factor $\gamma(v) \approx 16$, which is really huge. In fact, 2 μs of the muon's time corresponds to 32 μs Earth-time. In 32 μs, a particle of speed $v = 0.998c_0$ will travel a distance of 9.5 km! Hence, many muons should make it to the ground! On the other hand, from the muon's point of view, it does only live a mere couple of microseconds. However, the distance from the point of creation in the upper atmosphere to sea level is reduced by length contraction by a factor of $1/\gamma(v) = \frac{1}{16}$; that is, if the distance to the ground is 9.5 km, the muon will only 'feel' that the distance is 600 meters.

Physical experiments have confirmed that the observed rate of decay of muons do indeed depend on their speed; see, for instance, (Rossi & Hall, 1941). Many other experiments have also confirmed the predictions of special relativity. In fact, special relativity is used *daily* in applied physics and in every-day consumer electronics. Hence, today, special relativity is a very well-established theory.

We will make the spatial analogue of Definition NN:

[29] When cosmic-ray protons hit atmospheric atomic nuclei, pions are created. Very soon (within meters), these decay into muons and neutrinos. (Wikipedia contributors, 2011)

[30] If you plan to serve Coca-Cola at the party, please make sure that the temperature of this beverage does not exceed the upper limit of 4°C. [And, of course, it has to be served in liquid form at approx. 1 atm. atmospheric pressure.]

Definition NN

The *proper length* of an object is the length of the object as measured in the rest frame of the object.

3.3 The Lorentz Transformation

Let $\wp\mathcal{F}1$ and $\wp\mathcal{F}2$ be two physical frames of reference [think of the ground and a spaceship, respectively], with relative speed v, and choose two mathematical frames in standard configuration, $\mathcal{F}1 \in \wp\mathcal{F}1$ and $\mathcal{F}2 \in \wp\mathcal{F}2$.

We want to find the relativistic transformation that supersedes the Galilean transformation (↑); let's call it Λ. Thus, if (t, x, y, z) are the coordinates of some event relative to $\mathcal{F}1$, and if (t', x', y', z') are the coordinates of the very same event according to an observer in $\mathcal{F}2$, then

$$\Lambda(t, x, y, z) = (t', x', y', z').$$

Just like the case of the Galilean transformation, this transformation has to be linear, and it is given by

Theorem 10

The relativistic transformation $\Lambda: \mathbb{R}^4 \to \mathbb{R}^4$ of the coordinates of an event between two frames with relative speed v in standard configuration is

$$\begin{aligned} t' &= \gamma(v)\left(t - \frac{vx}{c_0^2}\right) \\ x' &= \gamma(v)(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

where $\gamma(v) = (1 - v^2/c_0^2)^{-1/2}$ is the Lorentz factor.

Proof

Let $\mathcal{F}2$ be a cubic spaceship with the origin somewhere on the rear side. Fix a point P inside the ship, at coordinates $(\tilde{x}', 0, 0)$ in $\mathcal{F}2$. Assume that a photon is emitted in the $\hat{\mathbf{x}}'$ direction at the common origin at time $t' = 0$, and that this photon has reached the detector at P at time $t' = \tilde{t}'$. Thus, in $\mathcal{F}2$, the coordinates of the detection event are $(\tilde{t}', \tilde{x}', 0, 0)$. Assume that the detection occurs at time $t = \tilde{t}$ in $\mathcal{F}1$. According to an $\mathcal{F}1$-bound observer, P is then located at position

$$\tilde{x} = v\tilde{t} + \frac{1}{\gamma(v)}\tilde{x}'$$

because the ship has travelled a distance $v\tilde{t}$ and the proper distance $\tilde{x}'$ between the emitter and detector inside the ship is contracted by a factor of $\gamma^{-1}(v)$. But this is a part in the sought transformation, namely

$$\tilde{x}' = \gamma(v)(\tilde{x} - v\tilde{t}).$$

Now[31], since $\mathcal{F}1$ moves with 'speed' $-v$ relative to $\mathcal{F}2$, the inverse transformation has to include

$$\tilde{x} = \gamma(v)(\tilde{x}' + v\tilde{t}').$$

Solve for $\tilde{x}'$ in this last equation and equate the expression with the forward transformation to obtain an equation in $\tilde{t}$ and $\tilde{t}'$ which is solved with respect to $\tilde{t}'$ to obtain

[31] At this point, it is tempting (wrongly) to 'deduce' $\tilde{t} = \gamma(v)\tilde{t}'$ since $\tilde{t}'$ is the 'lifetime' of the photon in $\mathcal{F}2$ and $\tilde{t}$ is the 'lifetime' in $\mathcal{F}1$, and there should be time dilation. By doing so, one 'derives' an incorrect 'Lorentz transformation'. Where lies the fault in this reasoning?

$$\tilde{t}' = \frac{1}{\gamma(v)v}\tilde{x} - \frac{\gamma(v)}{v}\tilde{x} + \gamma(v)\tilde{t} = \gamma(v)\tilde{t} - \frac{\gamma(v)v}{c_0^2}\tilde{x}$$

Therefore, since P was arbitrary, and since Λ 'has'[32] to map $y' = y$ and $z' = z$, we have shown that *any* $\mathcal{F}1$ coordinate quadruple (t, x, y, z) has the corresponding $\mathcal{F}2$ coordinates (t', x', y', z') given by these four relations, which thus is the sought transformation Λ. ■

A relativistic coordinate transformation between two inertial frames with coinciding spatial origin at the common origin of time is called a *Lorentz transformation*. The transformation of Proposition 6 thus is the Lorentz transformation in the case of standard configuration and relative speed v. Notice that, if $\mathcal{F}2$ moves with speed $+v$ relative to $\mathcal{F}1$, then $\mathcal{F}1$ moves with speed $-v$ relative to $\mathcal{F}2$. This means that the inverse transformation is obtained simply by changing v to $-v$ in Proposition 6.

> **Exercise:** Deduce the phenomena of time dilation and length contraction from the Lorentz transformation alone.

3.3.1 The Velocity Addition Formulae

In Newtonian physics, a particle moving with velocity $\mathbf{u}$ in $\mathcal{F}2$, which in turn moves with velocity $\mathbf{v}$ relative $\mathcal{F}1$ will be observed to move with the (geometric) velocity $\mathbf{v} + \mathbf{u}$ relative $\mathcal{F}1$. But then we saw that a light signal does not follow this law, which is frustrating. We will now derive the relativistic velocity addition formulae from the Lorentz transformation, and we will find that the (components of the) velocity of a photon transforms according to the very same set of formulae as ordinary matter, thus confirming the consistency, not to mention beauty, of special relativity.

Proposition 11

Let $\mathcal{F}1$ and $\mathcal{F}2$ be in standard configuration with relative speed v and denote by (u_x, u_y, u_z) the components of the velocity of some particle relative to $\mathcal{F}1$, and let (u'_x, u'_y, u'_z) be the components of the velocity relative to $\mathcal{F}2$. Then

$$u'_x = \frac{u_x - v}{1 - u_x v/c_0^2}$$
$$u'_y = \frac{u_y}{\gamma(v)(1 - u_x v/c_0^2)}$$
$$u'_z = \frac{u_z}{\gamma(v)(1 - u_x v/c_0^2)}$$

where $\gamma(v) = (1 - v^2/c_0^2)^{-1/2}$ is the Lorentz factor.

The asymmetry between the x direction on one hand and the y and z directions on the other comes from the choice of the standard configuration, where the relative velocity $\mathbf{v} = v\hat{\mathbf{x}}$.

[32] This is rather an assumption. The case is this: we have found that the Galilean transformation cannot be valid, and so we must device a new transformation that is valid. The most natural approach is to alter the Galilean transformation as little as possible and only tweak it so that all contradictions disappear. Experimental verification tells us that we need no more tweaks than this.

Proof

Differentiation of the Lorentz transformation (Theorem 10) yields

$$\begin{aligned} dt' &= \gamma(v)dt - \frac{v\gamma(v)}{c_0^2}dx \\ dx' &= \gamma(v)dx - v\gamma(v)dt \\ dy' &= dy \\ dz' &= dz \end{aligned}$$

and so

$$u_x' = \frac{dx'}{dt'} = \frac{\gamma(v)dx - v\gamma(v)dt}{\gamma(v)dt - \frac{v\gamma(v)}{c_0^2}dx} = \frac{dx - vdt}{dt - \frac{v}{c_0^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c_0^2}\frac{dx}{dt}} = \frac{u_x - v}{1 - \frac{v}{c_0^2}u_x},$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma(v)dt - \frac{v\gamma(v)}{c_0^2}dx} = \frac{\frac{dy}{dt}}{\gamma(v) - \frac{v\gamma(v)}{c_0^2}\frac{dx}{dt}} = \frac{u_y}{\gamma(v) - \frac{v\gamma(v)}{c_0^2}u_x}$$

and similarly with u_z'. ■

Example 12

Let $\mathcal{F}1$ be some inertial frame, and let $\mathcal{F}2$ be a spaceship travelling with speed v relative to $\mathcal{F}1$, and assume standard configuration.

1) Assume that $\mathcal{F}2$ fires a cannon ball in the $\hat{\mathbf{x}}'$ direction with speed u'. Let $v = 0.05c_0$ and $u' = 0.1c_0$. Classically, $\mathcal{F}1$ would observer the speed $u = v + u' = 0.15c_0 = 44970000$ m/s of the ball. Relativistically, however, we find $u = 44750000$ m/s using Proposition 11.

2) Assume that $\mathcal{F}2$ instead fires a photon (laser beam) in the $\hat{\mathbf{x}}'$ direction with speed (of course) $u' = c_0$. Classically, $\mathcal{F}1$ would observe the speed $u = v + u' = 1.05c_0$ of the photon. Using Proposition 11, however, we find $u = u' = c_0$, as expected.

Corollary 13

Let $\mathcal{F}1$ and $\mathcal{F}2$ be inertial frames with relative speed $v < c_0$ in standard configuration. Let $0 \leq u_x' \leq c_0$ be any speed as measured in $\mathcal{F}2$. Then $u_x \leq c_0$.

Proof

$$u_x = \frac{u_x' + v}{1 + u_x'v/c_0^2} \Rightarrow \frac{du_x}{du_x'} = c_0^2\frac{c_0^2 - v^2}{(c_0^2 + vu_x')^2} > 0, \qquad \forall u_x' \in [0, c_0]$$

so that $u_x' \mapsto u_x$ is a strictly increasing function. But $u_x(c_0) = c_0$ and the corollary follows. ■

3.3.2 The Acceleration Transformation Formulae

In Newtonian physics, $\mathbf{a}' = \mathbf{a}$ for any (geometric) acceleration vector. In particular, in standard configuration, even the components agree between the frames. In special relativity, we have

Proposition 14

Let $\mathcal{F}1$ and $\mathcal{F}2$ be in standard configuration with relative speed v and denote by (a_x, a_y, a_z) the components of the acceleration of some particle relative to $\mathcal{F}1$, and let (a'_x, a'_y, a'_z) be the components of the acceleration relative to $\mathcal{F}2$. Then

$$a'_x = \frac{1}{\gamma^3(v)(1-u_x\,v/c_0^2)^3}a_x$$
$$a'_y = \frac{1}{\gamma^2(v)\left(1-\frac{u_x v}{c_0^2}\right)^2}a_y + \frac{u_y v}{c_0^2\gamma^2(v)\left(1-\frac{u_x v}{c_0^2}\right)^3}a_x$$
$$a'_z = \frac{1}{\gamma^2(v)\left(1-\frac{u_x v}{c_0^2}\right)^2}a_z + \frac{u_z v}{c_0^2\gamma^2(v)\left(1-\frac{u_x v}{c_0^2}\right)^3}a_x$$

where $\gamma(v) = (1-v^2/c_0^2)^{-1/2}$ is the Lorentz factor.

Proof

Differentiation of the velocity transformation (Proposition 11) yields

$$du'_x = \frac{v}{c_0^2}(u_x - v)\left(1-\frac{u_x v}{c_0^2}\right)^{-2}du_x + \left(1-\frac{u_x v}{c_0^2}\right)^{-1}du_x = \frac{c_0^2 - v^2}{c_0^2\left(1-\frac{u_x v}{c_0^2}\right)^2}du_x =$$
$$= \frac{1}{\gamma^2(v)\left(1-\frac{u_x v}{c_0^2}\right)^2}du_x$$

It is still the case that

$$dt' = \gamma(v)dt - \frac{v\gamma(v)}{c_0^2}dx$$

and so

$$a'_x = \frac{du'_x}{dt'} = \frac{du_x}{\gamma^2(v)\left(1-\frac{u_x v}{c_0^2}\right)^2\left(\gamma(v)dt - \frac{v\gamma(v)}{c_0^2}dx\right)} = \frac{du_x/dt}{\gamma^3(v)\left(1-\frac{u_x v}{c_0^2}\right)^2\left(1-\frac{v}{c_0^2}\frac{dx}{dt}\right)} =$$
$$= \frac{a_x}{\gamma^3(v)\left(1-\frac{u_x v}{c_0^2}\right)^3}.$$

Next, we find the differential

$$du'_y = \gamma^{-1}(v)\frac{v}{c_0^2}u_y\left(1-\frac{u_x v}{c_0^2}\right)^{-2}du_x + \frac{du_y}{\gamma(v)(1-u_x v/c_0^2)};$$

thus

$$a'_y = \frac{du'_y}{dt'} = \frac{vu_y}{c_0^2\gamma^2(v)\left(1-\frac{u_x v}{c_0^2}\right)^2\left(dt-\frac{v}{c_0^2}dx\right)}du_x + \frac{du_y}{\gamma^2(v)(1-u_x v/c_0^2)\left(dt-\frac{v}{c_0^2}dx\right)} =$$

$$= \frac{vu_y}{c_0^2\gamma^2(v)\left(1-\frac{u_x v}{c_0^2}\right)^2\left(1-\frac{v}{c_0^2}\frac{dx}{dt}\right)}\frac{du_x}{dt} + \frac{\frac{du_y}{dt}}{\gamma^2(v)(1-u_x v/c_0^2)\left(1-\frac{v}{c_0^2}\frac{dx}{dt}\right)} =$$

$$= \frac{vu_y}{c_0^2\gamma^2(v)\left(1-\frac{u_x v}{c_0^2}\right)^3}a_x + \frac{a_y}{\gamma^2(v)(1-u_x v/c_0^2)^2}.$$

Of course, the z component is found in a perfectly symmetric manner. ■

3.3.3 Relativity of Simultaneity

Let $\mathcal{F}1 \in \wp\mathcal{F}1$ (the ground) and $\mathcal{F}2 \in \wp\mathcal{F}2$ (a ridiculously high-speed train) be in standard configuration with relative speed v. Assume that the train has proper length $\ell_t = 22$ m, and that it passes through a barn, stationary with respect to the ground, of proper length $\ell_b = 20\text{ m} < \ell_t$. Choose $v = 0.5c_0$. Then, according to an observer in $\wp\mathcal{F}1$, the length of the train is $\gamma^{-1}(v)\ell_t \approx 19$ m. That is, the train fits inside the barn, so that, for a femtosecond or so, when the rear end of the train has just moved inside the barn, the doors of the barn can close without even touching the train, which is then fully enclosed by the barn. The farmer does this, and it works.

However, from the perspective of an observer inside the train, the train has length $\ell_t = 22$ m, but the barn has shrunk to a mere $\gamma^{-1}(v)\ell_b \approx 17$ m. Thus, when the rear side of the train has just passed the left end of the barn, the front is already outside the right end of the barn, and so the doors cannot possible close at that time.

This is not a contradiction. Let the train start at $x' = 0$ and end at $x' = \ell_t$ in $\mathcal{F}2$. Similarly, let the barn start at $x = 0$ and end at $x = \ell_b$ in $\mathcal{F}1$. Then, at time $t = t' = 0$, the rear end of the train has just entered the barn. Thus, the event $\mathcal{E}1$ when the rear door is closed has coordinates $(0,0,0,0)$ in both systems. In $\mathcal{F}1$, the front door is shut at the same time as the rear door. Thus, the coordinates of the event $\mathcal{E}2$ of closing the front door are $(0,\ell_b,0,0)$ according to a $\mathcal{F}1$-bound observer, the farmer, say. But the Lorentz transformation, Theorem 10, then states that the coordinates of $\mathcal{E}2$, as seen by the train driver in $\mathcal{F}2$, are $(-c_0^{-2}v\gamma(v)\ell_b, \gamma(v)\ell_b, 0, 0) \approx (-39\text{ ns}, 23\text{ m}, 0, 0)$. That is, the front door is not closed at the same time as the rear door, the train driver finds! It is closed 'long' before the rear end of the train has entered the barn at $t' = 0$. In fact, when the front door is closed, it is located 23 meters ahead of the rear end of the train, that is, an entire metre in front of the train!

The above *Gedankenexperiment* exemplifies

Proposition 15

Let $\mathcal{F}1 \in \wp\mathcal{F}1$ and $\mathcal{F}2 \in \wp\mathcal{F}2$ be in standard configuration with relative speed v, and let $\mathcal{E}1$ and $\mathcal{E}2$ be events occurring simultaneous in $\mathcal{F}1$. Assume that $\mathcal{E}1$ has coordinates $(0,0,0,0)$ and that $\mathcal{F}2$ has coordinates $(\tilde{t},\tilde{x},\tilde{y},\tilde{z})$, where $\tilde{t} = 0$, all relative to $\mathcal{F}1$. Then $\mathcal{E}1$ and $\mathcal{E}2$ are simultaneous in $\mathcal{F}2$ if and only if $\tilde{x} = 0$ or $v = 0$.

Proof

Assume the hypotheses of the proposition. Then the coordinates of $\mathcal{E}1$, as seen in $\mathcal{F}2$, are also $(0,0,0,0)$. In particular, $\mathcal{E}1$ happens at time $t'=0$ in $\mathcal{F}2$. Let $\tilde{t}'$ denote the time coordinate of $\mathcal{E}2$ relative to $\mathcal{F}2$. Then $\mathcal{E}1$ and $\mathcal{E}2$ are simultaneous in $\mathcal{F}2$ if and only if $\tilde{t}'=t'$, that is, iff $\tilde{t}'=0$. As seen from the Lorentz transformation (Theorem 10), $\tilde{t}'=0 \Leftrightarrow \tilde{t}=v\tilde{x}/c_0^2$. But $\tilde{t}=0$ so that this is equivalent to $(\tilde{x}=0) \vee (v=0)$. ■

A well-known *Gedankenexperiment* illustrating the relativity of simultaneity directly from Einstein's postulates concerns a railway vehicle moving at high speed across a platform. At the very centre of the car, a person simultaneously emits one flash of light in the forward direction and one in the reverse direction. According to this person, the two walls of the car are hit by the light simultaneously. However, according to an observer at rest on the platform, the speed of light is still c_0 in *both* directions. Therefore, since the speed of light is finite, and the train is moving forwards, this observer will notice that rear side of the car is hit by the light before the front side is.

3.4 Dynamics in Special Relativity

In this section, we will discuss the dynamics of special relativity, particularly the concepts of momentum, force, and energy. In the next chapter, we will develop a more beautiful version of the dynamics using four-vectors on spacetime, but for now, we will still treat space and time in a more classical manner and rely only on three-component spatial vectors.

3.4.1 Rest Mass

By the 'mass' of an object, we mean the mass as measured in the rest frame of the object. Defined this way, the mass, also called the *invariant mass*, or the *rest mass*, is the same in every frame of reference. Self-evident or not, this is worth pointing out, since the related concept of *relativistic mass* is sometimes used instead, and this quantity is frame-dependant. We will return to this subject later.

3.4.2 Momentum

We will now investigate momentum in relativity. Of course, in any given inertial physical frame $\wp\mathcal{F}$, we could determine the rest mass and velocity of a particle (or system of particles), and compute $\sum m_i \mathbf{u}_i$. Then we could call this the 'momentum' of the particle (or system). However, the concept will only be useful in special relativity if it is a conserved quantity here as well. So, is it? In relativity, Newton's laws are not postulates, and so there is no *guarantee* that the quantity is conserved. In fact, we will show that, in general, it is *not*.

Let us consider the simplest example of an *inelastic* collision we can imagine. [There is no 'simple' example of an elastic collision that is interesting to us right now, for the reason given below.] Let A and B be two identical billiard balls of mass m with velocities $\mathbf{u}_A = u\hat{\mathbf{x}}$ and $\mathbf{u}_B = -u\hat{\mathbf{x}}$, respectively, about to collide at the origin, everything as seen from an inertial frame $\wp 1 \in \wp\mathcal{F}1$. We will use a tilde to denote a post-collision quantity. Thus the post-collision velocities are $\tilde{\mathbf{u}}_A = \tilde{\mathbf{u}}_B = \mathbf{0}$. Then, clearly, as seen from $\mathcal{F}1$, Newtonian momentum is conserved.

Now consider a different frame $\mathcal{F}2 \in \wp\mathcal{F}2$ in standard configuration relative to $\mathcal{F}1$ with relative speed $v = u$ equal to the pre-collision speed of A. In other words, $\mathcal{F}2$ is the pre-collision rest frame of A. According to Proposition NN, the $\mathcal{F}2$ pre-collision and post-collision velocities are

$$\mathbf{u}'_A = \mathbf{0}, \qquad \mathbf{u}'_B = -\frac{2u}{1+\frac{u^2}{c_0^2}}\hat{\mathbf{x}}'$$

and

$$\tilde{\mathbf{u}}'_A = -u\hat{\mathbf{x}}', \qquad \tilde{\mathbf{u}}'_B = -u\hat{\mathbf{x}}'.$$

Consequently, the total Newtonian momentum changes from

$$\sum m_i \mathbf{u}'_i = \mathbf{0} - \frac{2mu}{1+\frac{u^2}{c_0^2}}\hat{\mathbf{x}}' = -\frac{2mu}{1+\frac{u^2}{c_0^2}}\hat{\mathbf{x}}'$$

to

$$\sum m_i \tilde{\mathbf{u}}'_i = -mu\hat{\mathbf{x}}' - mu\hat{\mathbf{x}}' = -2mu\hat{\mathbf{x}}'.$$

Thus the absolute change

$$|\Delta\mathbf{p}'| \coloneqq \left|\sum m_i\tilde{\mathbf{u}}_i' - \sum m_i\mathbf{u}_i'\right| = 2mu\left(\frac{1}{1+\frac{u^2}{c_0^2}} - 1\right) \neq 0$$

and Newtonian momentum is *not* conserved in $\mathcal{F}2$. (Even though – of course – it is conserved to an excellent approximation at every-day speeds $u \ll c_0$.)

Now, this example is a bit special in that the collision is totally inelastic. Of course, special relativity does not forbid inelastic collisions, but in a later section we will see that 'strange' things happen in such collisions (when relativistic effects are taken into account). Perhaps Newtonian momentum *is* conserved in relativity as well, as long as we restrict our attention to elastic collisions? Obviously, such a restriction would seriously diminish the usefulness of the concept of 'momentum', but we will see next that not even in this very restrictive class of experiments is Newtonian momentum universally conserved. To show this, we *cannot* consider any of the simplest experimental setups for elastic collisions, in which two identical balls A and B have opposite pre-collision velocities $\mathbf{u}_A$ and $\mathbf{u}_B = -\mathbf{u}_A$ and collide at the origin at which point they reverse their velocities, so that $\tilde{\mathbf{u}}_A = -\mathbf{u}_A$ and $\tilde{\mathbf{u}}_B = -\mathbf{u}_B$. The reason is that, in such an experiment, the pre-collision situation is *identical* to the post-collision situation as far as velocities are concerned. Indeed, $\tilde{\mathbf{u}}_A = \mathbf{u}_B$ and $\tilde{\mathbf{u}}_B = \mathbf{u}_A$, so the *identical* balls merely swap velocities. Therefore, momentum will be conserved almost no matter how it is defined (e.g., the quantity $\sum m_i\mathbf{u}_i \sinh^2 u_i$ is conserved in this collision).

Therefore, we need to consider a slightly more complicated setup. One possibility is this: Let

$$\mathbf{u}_A = a\hat{\mathbf{x}} - b\hat{\mathbf{y}}, \qquad \mathbf{u}_B = b\hat{\mathbf{y}}$$

where $a, b > 0$. Thus, A is moving to the right and downwards, while B is moving straight upwards. Let them collide at the origin. Assume that Newtonian momentum is conserved; for instance, let the post-collision velocities be

$$\tilde{\mathbf{u}}_A = a\hat{\mathbf{x}} + b\hat{\mathbf{y}}, \qquad \tilde{\mathbf{u}}_B = -b\hat{\mathbf{y}}.$$

This collision is clearly allowed in Newtonian mechanics; indeed, it is head-on. Now let $\mathcal{F}2$ be in standard configuration relative to $\mathcal{F}1$ with relative speed $v = a$. In this frame, the pre-collision velocities are

$$\mathbf{u}_A' = -\frac{b}{\gamma(a)\left(1-\frac{a^2}{c_0^2}\right)}\hat{\mathbf{y}}', \qquad \mathbf{u}_B' = -a\hat{\mathbf{x}}' + \frac{b}{\gamma(a)}\hat{\mathbf{y}}'$$

while the post-collision velocities are

$$\tilde{\mathbf{u}}_A' = \frac{b}{\gamma(a)\left(1-\frac{a^2}{c_0^2}\right)}\hat{\mathbf{y}}', \qquad \tilde{\mathbf{u}}_B' = -a\hat{\mathbf{x}}' - \frac{b}{\gamma(a)}\hat{\mathbf{y}}'.$$

Therefore, the pre-collision total Newtonian momentum is

$$\sum m_i\mathbf{u}_i' = -\frac{mb}{\gamma(a)\left(1-\frac{a^2}{c_0^2}\right)}\hat{\mathbf{y}}' - ma\hat{\mathbf{x}}' + \frac{mb}{\gamma(a)}\hat{\mathbf{y}}'$$

and the post-collision momentum is

$$\sum m_i \tilde{\mathbf{u}}_i' = \frac{mb}{\gamma(a)\left(1-\frac{a^2}{c_0^2}\right)}\hat{\mathbf{y}}' - ma\hat{\mathbf{x}}' - \frac{mb}{\gamma(a)}\hat{\mathbf{y}}'.$$

Clearly, the x component of momentum happens to be conserved, but the y component changes from

$$\frac{mb}{\gamma(a)} - \frac{mb}{\gamma(a)\left(1-\frac{a^2}{c_0^2}\right)} \neq 0$$

(since $a, b, m > 0$) to

$$-\left(\frac{mb}{\gamma(a)} - \frac{mb}{\gamma(a)\left(1-\frac{a^2}{c_0^2}\right)}\right)$$

and so Newtonian momentum is not conserved in $\mathcal{F}2$. That is, not even in totally elastic collisions is the Newtonian law of momentum conservation valid. Clearly, we have to abandon the Newtonian momentum $\sum m_i \mathbf{u}_i$. Although momentum so defined is a universally conserved quantity in all every-day situations (to experimental accuracy), we have seen that strict universal conservation is impossible in general when relativistic effects are taken into account.

However, all is not lost. It turns out that there is quantity that is fundamentally conserved even in relativity theory, and that tends to the Newtonian expression for momentum in the Newtonian limit of low speeds. For this reason, we call this the 'relativistic momentum'.

Definition

The (relativistic) *momentum* of a particle of mass m and velocity $\mathbf{u}$ is

$$\mathbf{p} \coloneqq \gamma(u)m\mathbf{u}$$

where $\gamma(u)$ is the Lorentz factor.

Notice that $\mathbf{p} \to m\mathbf{u}$ as $u \to 0$, and – perhaps more importantly – that the relativistic momentum is equal to the Newtonian momentum to experimental accuracy if $u \ll c_0$. The (approximate) law of conservation of Newtonian momentum in the Newtonian limit $u \ll c_0$ thus becomes a corollary of the conservation of relativistic momentum.

And, indeed, relativistic momentum is believed to be a fundamentally conserved quantity in nature, even when full relativistic effects are considered. Unfortunately, there is no simple proof of this [or proof at all…], as there is in the Newtonian case. Thus, at a first glance it might seem that the relativistic law of momentum conservation is profoundly less well motivated than the Newtonian law, but this is not the case.

The reason why we can prove the Newtonian law of momentum conservation is that we happen to have a *postulate* in the Newtonian theory that fits like a glove for this purpose, namely, Newton's third law. Hence, the Newtonian law of momentum conservation follows immediate from a postulate of the theory, and so it is essentially a postulate itself. Furthermore, the only rationale for a postulate in a physical theory is that it agrees with observations. From the above it should

be clear that it is not a *deficiency* of the special theory of relativity if we need to postulate the conversation of (relativistic) momentum.

But if laws of momentum conservation cannot be proved, then how could we prove that Newtonian momentum is *not* conserved in relativity? Well, that is a different thing. You cannot *prove* a physical law of conversation unless it follows from the postulates of the theory. However, given a *proposed* law of conservation, you might actually be able to prove that this *cannot* be valid, by assuming that the law is valid one frame $\mathcal{F}1 \in \wp\mathcal{F}1$ and then use the appropriate coordinate transformation (Galilean or Lorentz, for instance), and see that the law is not valid in some other frame. If you can show that it is *not* valid in some other frame $\mathcal{F}2 \in \wp\mathcal{F}2$, then you have shown that the law cannot be valid. Indeed, not only does it reduce the applicability of a conservation law if it only applies in some frames, but it is also impossible to tell in which frames it is valid, for every inertial frame is equivalent to any other inertial frame. Consequently, if a law only applies in some inertial frames (at most), then it isn't a law of nature at all.

More succinctly put: if a proposed law of nature isn't compatible with the postulated coordinate transformation, then it isn't a law of nature.[33]

Using this technique, we have shown that the law of conversation of Newtonian momentum cannot be valid (other than approximately) in relativity, for it is not compatible with the Lorentz transformation. We also proposed an alternative law, the law of conversation of *relativistic* momentum. Although we cannot prove this, we can verify that it – contrary to the law of conservation of Newtonian momentum – *is* compatible with the Lorentz transformation. This we will do in a later section. In fact, we *cannot* prove it right now, because the proof requires an additional hypothesis, namely, the fact that also the 'relativistic energy' is conserved in the original frame (that is, in the frame in which we postulate the conservation of relativistic momentum). In other words, the concepts of momentum and energy seem to be 'entangled' in special relativity.

Anyhow, we now state

Postulate NN

The total (relativistic) momentum is a fundamentally conserved quantity in any isolated system.

By 'total relativistic momentum, we mean (at least so far) the sum $\sum_{i\in X}\gamma(u_i)m_i\mathbf{u}_i$ if the isolated system consists only of a set of discrete matter particles with index set X. We do not yet dare to speak about kinds of systems. Also, from now on, by 'momentum' we mean *relativistic* momentum.

3.4.3 Proper Force

What do we mean by 'force'? Classically, if an object is found to have acceleration $\mathbf{a}$ in an inertial frame, and is known to have (rest) mass m, we deduce that the net force on the object is $\mathbf{F} = m\mathbf{a}$. Now, in relativity theory, since in any frame $\mathcal{F}$ we *can* measure the components of the acceleration, and we know the (rest) mass of the object, we *could* easily compute the components of $m\mathbf{a}$ and call that the 'force' on the object. But we will see that this isn't a helpful concept.

Assume that a particle travels along the x axis of some inertial system $\mathcal{F}1 \in \wp\mathcal{F}1$. The acceleration is measured to be a, and so the 'force' is deduced to be

[33] In this terminology, we proved in Proposition NN that the law of conservation of *Newtonian* momentum is compatible with the *Galilean* transformation.

$$F = ma.$$

Now, let $\mathcal{F}2 \in \wp\mathcal{F}2$ be another frame in standard configuration relative to $\mathcal{F}1$ and with relative speed v. In this frame, the acceleration is found to be

$$a' = \frac{1}{\gamma^3(v)(1 - u_x\, v/c_0^2)^3}a$$

according to Proposition 14. Hence, a $\mathcal{F}2$-bound observer will deduce that the 'force' on the particle is

$$F' = \frac{ma}{\gamma^3(v)(1 - u_x\, v/c_0^2)^3} \neq ma$$

(unless $v = 0$, $v = u_x$, or $a = 0$). This is in contrast with the Newtonian case, where the acceleration (geometrically) is the same in any inertial frame, and so the force is the same; in standard configuration, even the components agree. It also entirely contradicts the usual idea that the force itself is a geometric, frame-independent, object. Intuitively, we would like to define some 'proper force' as the product of mass and acceleration as measured in the rest frame of the object. This way the 'proper force' would be the same in any frame of reference. One problem with this approach, however, is that the rest frame of the object is not an inertial frame if the object is accelerating, and so we do not know how to work in it. For example, Einstein's postulates only concerns inertial frames. We will avoid this difficulty by doing some 'trickery'.

Definition 17

Let A be any object travelling through space. At any proper time t as measured by a clock travelling with A, let $\mathcal{IRF}_t$, the *instantaneous physical rest frame* of A at time t, be the physical frame containing all *inertial* mathematical frames relative to which A is stationary (that is, momentarily at rest) at time t. Any mathematical frame $\mathcal{F} \in \mathcal{IRF}_t$ is called an *instantaneous rest frame* (IRF) of A at time t. A *standard* instantaneous rest frame is an instantaneous rest frame $\mathcal{F} \in \mathcal{IRF}_t$ such that A is located at the origin of $\mathcal{F}$ at time t.

Notice that for every t, every $\mathcal{F} \in \mathcal{IRF}_t$ is an inertial frame, and so we can do computations in it. If A travels with constant velocity relative to some inertial frame, then $\mathcal{IRF}_{t_1} = \mathcal{IRF}_{t_2}$ for every pair (t_1, t_2). If, on the other hand, A has a non-zero acceleration at a time t, there exists an $\epsilon > 0$ such that $\mathcal{IRF}_t \neq \mathcal{IRF}_{t+\Delta t}$ for all $0 < |\Delta t| < \epsilon$. [Assuming that the acceleration is a continuous function of proper time.]

Now we are ready to talk about forces. We will start with *one* 'imaginable' definition of the concept:

Definition 18

Let A be any object travelling through space. The *proper acceleration* $\mathbf{a}_0$ on A at time t is the acceleration as measured in an instantaneous rest frame $\mathcal{F} \in \mathcal{IRF}_t$. The *proper force* $\mathbf{F}_0$ on A at time t is the product of rest mass and proper acceleration at this time, that is,

$$\mathbf{F}_0 = m\mathbf{a}_0.$$

Notice that it doesn't matter *which* mathematical instantaneous rest frame we use to measure the acceleration, since it is a geometric vector independent of the coordinate system employed *inside* any given physical frame. This definition resembles the definition of the rest mass. In that

discussion, we said something like "defined this way, the rest mass is the same in any inertial frame of reference". By that, we mean that *any* inertial observer can confirm that the rest mass has a particular value, m say, simply by moving to an (instantaneous) rest frame and measuring the mass. The situation is more complicated here (when talking about the proper force), because the proper force is a *vector*, and in order to specify a vector (using its components), you need to agree on a basis, and there is generally no obvious choice of such a basis in the IRF. Nevertheless, there is nothing wrong with Definition 18 *per se*, but one needs to realise that it is awkward to talk about the proper force if you do not live inside an IRF of the object being under consideration.

Assume that you live inside a physical frame $\wp\mathcal{F}1$, and that you are observing an object A with velocity $\mathbf{u}$ and (physical) instantaneous rest frame $\mathcal{IRF}_t$ at some particular instance of time. Assume that the object is not at rest, that is, assume that $\wp\mathcal{F}1 \neq \mathcal{IRF}_t$. We are going to agree on a choice of mathematical frames $\mathcal{F}1 \in \wp\mathcal{F}1$ and $\mathcal{F}2 \in \mathcal{IRF}_t$ in each of these two physical frames. We require that

- $\mathcal{F}2$ is a *standard* IRF, and that
- $\mathcal{F}1$ and $\mathcal{F}2$ are in standard configuration.

Since $\mathcal{F}2$ is an IRF of A, the origin of $\mathcal{F}2$ is moving with velocity $\mathbf{v} = \mathbf{u}$ relative to $\mathcal{F}1$, where $\mathbf{u}$ is the velocity of A relative to $\mathcal{F}1$. But the second requirement above implies $\mathbf{v} = v\hat{\mathbf{x}}$. Thus $\mathbf{u} = u\hat{\mathbf{x}}$, that is, A is moving (momentarily, at least) along the x axis at the chosen instance of time. Pick *any* convenient spatial (P) and temporal origin of $\mathcal{F}1$. Then we have determined the mathematical frames $\mathcal{F}1$ and $\mathcal{F}2$ up to a rotation of the basis vectors about the $\hat{\mathbf{x}}$ direction.

From now on, every time we talk about a vector in $\wp\mathcal{F}1$ or $\mathcal{IRF}_t$, the components are understood to be relative to the respective bases.

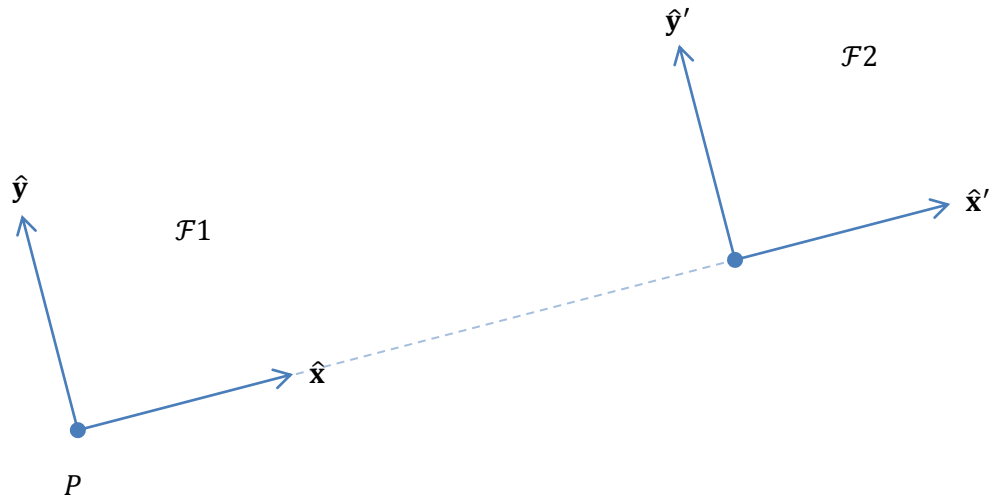


Figure 30. An inertial frame and an instantaneous rest frame.

Now, let's start doing some calculations. Let the components of the geometric proper acceleration vector $\mathbf{a}_0$ be $(a_{0,x}, a_{0,y}, a_{0,z})$ relative to the basis of $\mathcal{F}2$. Then the components of the acceleration with respect to $\mathcal{F}1$ are

$$a_x = \gamma(v)^{-3} a_{0,x}$$
$$a_y = \gamma(v)^{-2} a_{0,y}$$

$$a_z = \gamma(v)^{-2} a_{0,z}$$

as deduced by applying [the inverse of the transformation in] Proposition 14 [since $(u'_x, u'_y, u'_z) = \mathbf{0}$]. Here v, the relative speed between $\mathcal{F}1$ and $\mathcal{F}2$, is equal to the speed of the object as measured in $\mathcal{F}1$. Let us therefore call it u instead. Multiplication by the (rest) mass m then yields

$$\begin{aligned} ma_x &= \gamma(u)^{-3} m a_{0,x} = \gamma(u)^{-3} F_{0,x} \\ ma_y &= \gamma(u)^{-2} m a_{0,y} = \gamma(u)^{-2} F_{0,y} \\ ma_z &= \gamma(u)^{-2} m a_{0,z} = \gamma(u)^{-2} F_{0,z} \end{aligned}$$

where $(F_{0,x}, F_{0,y}, F_{0,z})$ are the components of the proper force. Hence,

$$\begin{aligned} F_{0,x} &= \gamma(u)^3 m a_x \\ F_{0,y} &= \gamma(u)^2 m a_y \\ F_{0,z} &= \gamma(u)^2 m a_z. \end{aligned}$$

In the special case of motion along a straight line, we have

Proposition 19

Assume that a particle is travelling along the x axis of some frame $\mathcal{F}1 \in \wp\mathcal{F}1$ and that, at some point, has speed u and acceleration a relative to $\mathcal{F}1$. Then

$$F_0 = \gamma(u)^3 m a$$

where F_0 is the proper force on the particle at that time.

We conclude this subsection by a beautiful relation. This far we have found a new, relativistic, definition of momentum, and, in addition, introduced the concept of 'proper force' as the force observed in an IRF. 'Accidentally', these definitions imply

Proposition 22

Let $\mathcal{F}$ be a frame and A an object travelling along the x axis of $\mathcal{F}$. Suppose that the proper force on A is F_0 and that, as measured in $\mathcal{F}$, the momentum of A is p. Then

$$F_0 = \frac{dp}{dt}.$$

Thus, the one-dimensional momentum-derivative form of Newton's second law has the exact same form in special relativity! Notice that the scalar proper force F_0 is independent of the frame of reference, whereas both the coordinate time t and momentum p depend on the frame $\mathcal{F}$. Apparently, these dependencies cancel as to make dp/dt frame-independent.

Proof of Proposition 22

According to Proposition 19, $F_0 = \gamma^3(u) m a$. On the other hand, Definition 16 states that $p = \gamma(u) m u$ so that

$$\frac{dp}{dt} = m\left(\frac{d\gamma(u)}{du}\frac{du}{dt}u + \gamma(u)\frac{du}{dt}\right) = m\left(\frac{d\gamma(u)}{du}u + \gamma(u)\right)a = m\gamma^3(u)a = F_0$$

since the rest mass m is a constant. ■

3.4.4 The Relativistic Force

It is natural to ask what the meaning of the *vector* $d\mathbf{p}/dt$ (which certainly is well-defined) is inside any given frame. Of course, *a priori*, we have to assume that this quantity is frame-dependant. We have already seen that, in the special case where the particle is confined to the x axis, $dp/dt = F_0$, and so, in this case,

$$\frac{d\mathbf{p}}{dt} = \left(\frac{dp_x}{dt}, \frac{dp_y}{dt}, \frac{dp_z}{dt}\right) = \left(\frac{d}{dt}(\gamma(u)mu_x), \frac{d}{dt}(\gamma(u)mu_y), \frac{d}{dt}(\gamma(u)mu_z)\right) = (F_0, 0, 0)$$

since the functions $t \mapsto u_x(t)$ and $t \mapsto u(t)$ are identical, and since $u_y = u_z = 0$ at all times. We will now investigate the general case, where the particle is *not* restricted to a straight line. In full generality,

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma(u)m\mathbf{u}) = m\left(\frac{d\gamma(u)}{dt}\mathbf{u} + \gamma(u)\frac{d\mathbf{u}}{dt}\right) = m\frac{\gamma(u)^3}{c_0^2}u\frac{du}{dt}\mathbf{u} + m\gamma(u)\mathbf{a}.$$

It may come as a surprise, but in general (and this is nothing else than Newtonian mechanics),

$$\frac{du}{dt} \neq a$$

where $u \overset{\text{def}}{=} |\mathbf{u}|$ and $a \overset{\text{def}}{=} |\mathbf{a}| \overset{\text{def}}{=} |d\mathbf{u}/dt|$.[34] Therefore, in general,

$$\mathbf{u} \cdot \mathbf{a} \neq u\frac{du}{dt}\cos\angle(\mathbf{u}, \mathbf{a}).$$

Instead,

$$\frac{du}{dt} = \frac{d}{dt}|\mathbf{u}| = \frac{d}{dt}\sqrt{u_x^2 + u_y^2 + u_z^2} = \frac{u_x a_x}{\sqrt{u_x^2 + u_y^2 + u_z^2}} + \frac{u_y a_y}{\sqrt{u_x^2 + u_y^2 + u_z^2}} + \frac{u_z a_z}{\sqrt{u_x^2 + u_y^2 + u_z^2}} = \frac{1}{u}\,\boldsymbol{u} \cdot \boldsymbol{a};$$

thus

$$\mathbf{u} \cdot \mathbf{a} = u\frac{du}{dt}.$$

Using this identity, (↑) reads

$$\frac{d\mathbf{p}}{dt} = m\frac{\gamma(u)^3}{c_0^2}(\mathbf{u} \cdot \mathbf{a})\mathbf{u} + m\gamma(u)\mathbf{a}.$$

Let

$$\mathbf{a} = \mathbf{a}_\parallel + \mathbf{a}_\perp$$

where the *parallel acceleration*

$$\mathbf{a}_\parallel \coloneqq \frac{(\mathbf{a} \cdot \mathbf{u})}{|\mathbf{u}|^2}\mathbf{u} \parallel \mathbf{u}$$

and the *orthogonal acceleration*

[34] Think of the case of circular motion with constant speed (cf. 1.4.3). Here $\mathbf{u}$ is constantly changing. Indeed, $\mathbf{a} \neq \mathbf{0}$ is constant and points to the centre of the orbit. However, the speed $u = |\mathbf{u}|$ is constant; thus $du/dt = 0$. On the other hand, since $\mathbf{a} \neq \mathbf{0}$, the vector length $a = |\mathbf{a}| = 0$. Hence $du/dt \neq a$.

$$\mathbf{a}_\perp \coloneqq \mathbf{a} - \mathbf{a}_\parallel \perp \mathbf{u}.$$

Then

$$\frac{d\mathbf{p}}{dt} = m\frac{\gamma(u)^3}{c_0^2}u^2\mathbf{a}_\parallel + m\gamma(u)(\mathbf{a}_\parallel + \mathbf{a}_\perp) = \left(m\frac{\gamma(u)^3}{c_0^2}u^2 + m\gamma(u)\right)\mathbf{a}_\parallel + m\gamma(u)\mathbf{a}_\perp =$$
$$= m\gamma(u)^3\mathbf{a}_\parallel + m\gamma(u)\mathbf{a}_\perp.$$

We now make

Definition NN

The *relativistic force* on a particle is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

where $\mathbf{p}$ is the (relativistic) momentum of the particle

and conclude

Proposition NN

Let $\mathbf{F}$ be the relativistic force on a particle with mass m, velocity $\mathbf{u}$, and acceleration $\mathbf{a}$. Then

$$\mathbf{F} = m\frac{\gamma(u)^3}{c_0^2}(\mathbf{u}\cdot\mathbf{a})\mathbf{u} + m\gamma(u)\mathbf{a} =$$
$$= m\gamma(u)^3\mathbf{a}_\parallel + m\gamma(u)\mathbf{a}_\perp$$

where $\mathbf{a} = \mathbf{a}_\parallel + \mathbf{a}_\perp$, $\mathbf{a}_\parallel \parallel \mathbf{u}$, and $\mathbf{a}_\perp \perp \mathbf{u}$.

3.4.5 Comparison between Proper Force and Relativistic Force

Assume the setup of Figure 30, where $\mathcal{F}2$ is an IRF of a particle that *momentarily* is moving in the $\hat{\mathbf{x}}$ direction relative to $\mathcal{F}1$. Let $\mathbf{F}_0 = (F_{0,x}, F_{0,y}, F_{0,z})$ be the proper force and let $\mathbf{F} = (F_x, F_y, F_z)$ be the relativistic force relative to $\mathcal{F}1$. Let the acceleration of the particle be

$$\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}} = \mathbf{a}_\parallel + \mathbf{a}_\perp$$

relative to $\mathcal{F}1$. Then the proper force components are

$$F_{0,x} = \gamma(u)^3 m a_x$$
$$F_{0,y} = \gamma(u)^2 m a_y$$
$$F_{0,z} = \gamma(u)^2 m a_z$$

(cf. ↑) while the relativistic force components are

$$F_x = \gamma(u)^3 m a_x$$
$$F_y = \gamma(u) m a_y$$
$$F_z = \gamma(u) m a_z$$

using Proposition NN. Clearly the proper force and the relativistic force are two *different* concepts. In addition, since the components of the proper force are Lorentz scalars (frame-independent), it is clear that the components of the relativistic force are *not*, that is, they *do* de-

pend upon the frame of reference.[35] The x component, however, coincides with the x component of the proper force. Thus

Observation NN

The proper force and the relativistic force are two *different* concepts. However, in the special case of rectilinear motion, they coincide. In particular, while the components of the proper force are Lorentz scalars, the components of the relativistic force are not.

Of course, an IRF of an object is equivalent to any other inertial frame, and so we can compute the relativistic force inside this frame. Since the object is at rest relative to this frame, $u = 0$ which implies $\gamma(u) = 1$ and so the relativistic force is

$$F_x = ma_x$$
$$F_y = ma_y$$
$$F_z = ma_z$$

which, by definition, are the components of the proper force. Thus

Observation NN

Inside an IRF of an object, the relativistic force on the object equals the proper force on it.

We should also state the obvious:

Observation NN

Both the proper and the relativistic force tends to the Newtonian force in the Newtonian (low-speed) limit, where the 'Newtonian force' is defined as the product of (rest) mass and acceleration.

Finally, we have to point out, once again, that we have merely *defined* two quantities, the proper force and the relativistic force. That's it. In particular, this means that, every time we talk about a 'force', we need to specify what we mean by 'force' in that case.

3.4.6 Constant Force

Not only is the concept of 'proper force' convenient because it is frame-independent, but it is also a very *natural* definition of the 'force' in general. To appreciate this, think of a spaceship that is accelerating. At any time, there is an instantaneous rest frame of the ship. It is relative to this frame that the engines operate. [Which other frame could it possibly be?] Therefore, if the en-

[35] This follows from the definition of the proper force, and the difference between the expressions (↑) and (↑). Still, these claims can be verified by far more naïve methods. Indeed, let $\mathcal{F}1$ and $\mathcal{F}2$ be two frames in standard configuration with *any* relative speed v, and choose the velocity (u_x, u_y, u_z) of a particle relative to $\mathcal{F}1$ and the acceleration (a_x, a_y, a_z), also relative to $\mathcal{F}1$. Compute the numbers $(F_{x,0}, F_{y,0}, F_{z,0}) = (\gamma(u)^3 ma_x, \gamma(u)^2 ma_y, \gamma(u)^2 ma_z)$ of the proper force. Now, use the Lorentz transformations to determine the components (u'_x, u'_y, u'_z) and (a'_x, a'_y, a'_z) of the velocity and acceleration, respectively, with respect to $\mathcal{F}2$. Then compute the proper force $(F'_{x,0}, F'_{y,0}, F'_{z,0}) = (\gamma(u')^3 ma'_x, \gamma(u')^2 ma'_y, \gamma(u')^2 ma'_z)$ in $\mathcal{F}2$. You should get $(F_{x,0}, F_{y,0}, F_{z,0}) = (F'_{x,0}, F'_{y,0}, F'_{z,0})$ for every choise of the parameters v, (u_x, u_y, u_z), and (a_x, a_y, a_z). This confirms that the proper force is a Lorentz invariant vector, when computed using (↑). Then redo the same calculations using the relativistic force instead of the proper force, using the similar, but different, expressions in Proposition NN. You will find that the components, except for the x component, depend upon the frame.

gines have a fixed power P, they will produce a constant force F relative to the $\mathcal{IRF}_t$. This is the 'proper force'.

However, as we saw in the last section, another 'reasonable' concept of force is the relativistic force. In this section, we will investigate the case of a constant force acting on an object. Hence, it might seem like we have to choose either one of the concepts. Indeed, our goal is to find the acceleration, and, via integration, the velocity and displacement formulae. Thus, we need to know if it is the proper force

$$\left(\gamma(u)^3 ma_x, \gamma(u)^2 ma_y, \gamma(u)^2 ma_z\right)$$

or the relativistic force

$$\left(\gamma(u)^3 ma_x, \gamma(u) ma_y, \gamma(u) ma_z\right)$$

that is constant. Fortunately, however, we will only be interested in long-distance, galactic travel, and such journeys occur along straight lines. Hence, we will only be moving along the x axis of some inertial coordinate system, and – lucky us – the x components agree. 'Problem solved!'

Now, let us investigate this more thoroughly. Let the constant (proper, say) force on the ship be F. Then, according to an observer in an inertial frame $\mathcal{F}1$ that is in standard configuration with every $\mathcal{IRF}_t \in \mathcal{IRF}$, the coordinate acceleration is

$$a = \frac{F_0}{\gamma^3(u)m}$$

from Proposition 19. This is

$$\frac{du}{dt} = \frac{F_0}{m}\left(1 - \frac{u^2}{c_0^2}\right)^{3/2},$$

that is, a first-order non-linear ODE in the coordinate speed u. It is clearly separable, for

$$f(u)\frac{du}{dt} \coloneqq \left(1 - \frac{u^2}{c_0^2}\right)^{-3/2}\frac{du}{dt} = \frac{F_0}{m}.$$

Integration yields

$$u\gamma(u) = \frac{F_0}{m}t + C$$

for some constant C, as is verified by differentiation. Assume that the ship starts from rest at the Earth (the spatial origin) at time $t = 0$; then $C = 0$. Solve for $u(t)$ and find

$$u(t) = \frac{F_0 t}{\sqrt{m^2 + c_0^{-2}F_0^2 t^2}}$$

assuming all quantities positive. This is trivial to integrate:

$$x(t) = \frac{c_0^2}{F_0}\sqrt{m^2 + c_0^{-2}F_0^2 t^2} + D.$$

Recalling that $F_0 = ma_0$, where a_0 is the proper acceleration, we find

$$x(t) = \frac{c_0}{a_0}\sqrt{c_0^2 + a_0^2 t^2} - \frac{c_0^2}{a_0}.$$

where we also have chosen $D = -c_0^2/a_0$ as is required by $x(0) = 0$. The inverse relation is

$$t(x) = \sqrt{\frac{x^2}{c_0^2} + \frac{2x}{a_0}}.$$

For future use, we also notice that (↑) and (↑) combine to give

$$u(x) = \sqrt{\frac{a_0^2x^2 + 2c_0^2a_0x}{c_0^2 + \frac{a_0^2x^2}{c_0^2} + 2a_0x}}.$$

Let us now pause for a few observations.

Observation 20

As seen from an inertial frame $\mathcal{F}$, the speed of an object accelerating along the x axis due to a constant proper force can never exceed the speed of light, but will approach it indefinitely.

Proof

As seen from $\mathcal{F}$, the speed of the object is

$$u(t) = \frac{F_0t}{\sqrt{m^2 + c_0^{-2}F_0^2t^2}} = \frac{1}{\sqrt{\frac{m^2}{F_0^2t^2} + \frac{1}{c_0^2}}}\begin{cases} < c_0 & \forall t > 0 \\ \to c_0 & \text{as } t \to \infty \end{cases}$$

■

We are now interested in how an observer on board the spaceship perceives the journey. Due to time dilation/length contraction, an on-board passenger should not think that the journey is as long as an observer at rest on Earth would. Let dt be a short interval of Earth time, and $d\tau$ the corresponding short moment of proper time inside the ship [by which we really mean inside an instantaneous rest frame]. Then $dt = \gamma(v)d\tau$ where v is the current speed of the ship. Let $t = \tilde{t}$ be the time of the arrival at Vega, say, relative to Earth, and let $t' = \tilde{t}'$ be the proper time inside the ship of the very same event. $t = t' = 0$ at Earth. Then

$$\tilde{t}' = \int_0^{\tilde{t}} d\tau = \int_0^{\tilde{t}} \frac{1}{\gamma(v)}dt = \int_0^{\tilde{t}} \sqrt{1 - \frac{v^2}{c_0^2}}dt = \int_0^{\tilde{t}} \sqrt{1 - \frac{F_0^2t^2}{c_0^2m^2 + F_0^2t^2}}dt$$

using (↑). Some algebra yields

$$\tilde{t}' = \int_0^{\tilde{t}} \frac{1}{\sqrt{1 + \frac{a_0^2}{c_0^2}t^2}}dt = \frac{c_0}{a_0}\operatorname{arcsinh}\left(\frac{a_0}{c_0}\tilde{t}\right).$$

Example 21

The (proper) distance between the Earth and Vega is $x = 25$ light-years. We assume a constant proper acceleration of $a_0 = 9.8\ \mathrm{m/s^2}$ which is optimal for the human musculoskeletal system. Then (↑) gives the duration $\tilde{t} = 26$ years. But a passenger will (according to herself) be there already in $\tilde{t}' = 3.9$ years.

Armed with the tremendous power of special relativity, we should really not be this modest. Let us instead travel to the Andromeda galaxy. The proper distance is now $x = 2.5$ million light-years. This journey will take 2.5 million years of Earth time. But a passenger will (again, according to herself) be there in only 15 years!

The diameter of the observable universe is $x = 93$ billion light-years. A journey of this distance takes 93 billion years, but a passenger will only age 25 years.

> **Exercise:** The above computations all assume that the traveller accelerates with constant proper acceleration the entire journey. A more realistic case is that the journey ends with zero speed. Assuming that the spaceship accelerates during the first half of the journey and slows down during the second half, how long would the above journeys take?

3.4.7 Energy

In addition to the kinematical surprises [an upper speed limit, time dilation, length contraction, relativity of simultaneity, etc.] that follow immediately from Einstein's postulates, a major 'result' in Special Relativity is the often-misunderstood 'equivalence' between mass and energy, as quantified by the easy-to-write but hard-to-explain relation $E = mc_0^2$. This is indeed a subject surrounded by great confusion. It is easy to think that one 'understands' the 'equivalence' just because one has memorized the extremely well-known formula $E = mc_0^2$, but this is utterly wrong. A formula is nothing without a context, and, in this case, the context is far from 'extremely well-known'.

First of all, the context need to explain what E and m really stand for in the formula. When it comes to m, from a classical point of view, we might want to specify what kind of mass we are talking about: the inertial mass, the active gravitational mass, or the passive gravitational mass. I do not think this is the main issue, because these masses are identical in essentially all established theories. Yet m is not unproblematic, because we need to specify if m is the rest mass or the relativistic mass. Even worse, if we start to speak about the mass of non-material fields, such as the 'mass' of an EM field, as indeed one often does when one speaks about the mass-energy equivalence, then what in the world do we mean by 'mass'?

When we have settled for a definition of m, we need to decide on what E quantifies. Is it the kinetic energy? The total mechanical energy? The *total* energy? In fact, just as we defined a new expression for the momentum in special relativity, we will define a new expression for the energy, both kinetic and total. There are several ways of doing this, and, whatever way we choose, we have to remember that we have only made a definition. We have postulated an expression (motivated by some rather weak ideas), and given it a special name. *A priori*, there is no reason to believe that this quantity should have any physical significance. It is interesting, however, that one of the most natural 'weak ideas' actually gives rise to the famous $E = mc_0^2$ formula rather quickly. In this way, special relativity 'suggests' the mass-energy 'equivalence', but it says nothing about its physical meaning, at least not when derived in the usual way, from the 'weak idea'.

Finally, if we have somehow settled for a precise definition of m and E, we need to specify *under what circumstances* $E = mc_0^2$ holds. Overall, $E = mc_0^2$ is easy-to-state but very hard to explain. Let us now go into the details. We will start by deriving $E = mc_0^2$ from the usual 'weak idea'.

We first define relativistic *work* on a particle as the line integral of proper force. For simplicity, we will restrict the analysis to the one-dimensional case. Therefore, you can equally well think of the proper force as being the relativistic force, if you prefer so. The spatial variables used in the line integral are *not* those of an instantaneous rest frame of the particle, but rather those of an observer in a frame relative to which we are interested in the 'relativistic kinetic energy' [which we define as the relativistic work required to obtain it]. Let the position of the particle be $x(t)$ at time t. Then the relativistic work done between $t = 0$ and $t = \tilde{t}$ is

$$\begin{aligned} W \coloneqq \int_0^{\tilde{t}} F_0 dx = \int_0^{\tilde{t}} \frac{d}{dt}(\gamma(u)mu)udt = m\int_0^{\tilde{t}} \left(u\frac{d}{dt}\gamma(u) + \gamma(u)a\right)udt = \\ = m\int_0^{\tilde{t}} \left(u\cdot\frac{u\gamma^3(u)}{c_0^2}a + \gamma(u)a\right)udt = m\int_0^{\tilde{t}} \left(\frac{u^2\gamma^2(u)}{c_0^2} + 1\right)\gamma(u)audt = \\ = m\int_0^{\tilde{t}} \gamma^3(u)audt. \end{aligned}$$

Restrict attention to the schoolbook case where $t \mapsto u(t)$ is a strictly increasing function. Then we can change variable of integration from t to u. Assume also that $u(0) = 0$ and $u(\tilde{t}) =: \tilde{u}$. Since $du = adt$,

$$W = m\int_0^{\tilde{u}} \gamma^3(u)udu = [m\gamma(u)c_0^2]_0^{\tilde{u}} = m\gamma(\tilde{u})c_0^2 - mc_0^2.$$

This motivates[36]

Definition 23

The *relativistic kinetic energy* of a particle of (rest) mass m and speed u is

$$E_k = \gamma(u)mc_0^2 - mc_0^2 = (\gamma(u) - 1)mc_0^2.$$

Taylor expansion yields

$$\gamma(u)mc_0^2 = mc_0^2 + \frac{1}{2}mu^2 + O(u^4)$$

and so we have

Proposition 24

Let a particle of rest mass m and speed u have relativistic kinetic energy E_k. Then

$$E_k \sim \frac{1}{2}mu^2, \qquad u \ll c_0.$$

Thus, the Newtonian kinetic energy is the low-speed limit of the relativistic kinetic energy. The expression in Definition 23 looks a bit funny. It 'sort of' suggests

Definition 24

The *total energy* of a particle of (rest) mass m and speed u is

[36] Notice that, although the speed of a particle is bounded from above by the speed of light, there is no limit on the kinetic energy; indeed, $E_k \to \infty$ as $u \to c_0$.

$$E = \gamma(u)mc_0^2.$$

The *rest energy* of such a particle is

$$E_0 = mc_0^2.$$

Notice that $E = E_0 + E_k$. In the rest frame of the particle, $u = 0$ and so $E_k = 0$ and $E = E_0$. That is, the rest energy is the total energy of the particle in its rest frame, where it has no kinetic energy. In a frame in which the particle is moving, the total energy is the sum of its rest energy and its kinetic energy. Hence, the rest energy is an invariant scalar – it is the same in every frame. The subscript zero reminds us of this. It is not uncommon that texts on special relativity at this point says something seemingly stupid like "thus, Special Relativity requires an object at rest to have a non-zero energy equal to mc_0^2". This is nonsense. First, Definition 24 is nothing but a definition, so the $E_0 = mc_0^2$ 'result' is a definition, not a theorem. In addition, we have no *a priori* reason to believe that the 'energy' defined by Definition 24 has any physical significance, such as conservation. In addition, from a *classical* (or, rather, Newtonian) point of view, should there be any rest energy, it should not be physically detectable in any way, for it is a constant, namely, mc_0^2, and we are always free to shift the zero of potential energy without altering any of the physics.

From now on, when we talk about total energy and kinetic energy, we mean total *relativistic* energy and *relativistic* kinetic energy, respectively. We also postulate

Postulate

The total (relativistic) energy is a fundamentally conserved quantity in any isolated system.

By 'total relativistic energy', we mean (at least so far) the sum $\sum_{i\in X}\gamma(u_i)m_ic_0^2$ if the isolated system consists only of a set of discrete matter particles with index set X. We do not yet dare to speak about kinds of systems.

3.4.8 Relativistic Mass

Before continuing our investigation of the physics of special relativity, we will introduce a special name for the product $\gamma(u)m$ between the Lorentz factor and the mass of a particle of mass m and speed u.

Definition 26

Let $\mathcal{F}$ be an inertial frame. The *relativistic mass* in $\mathcal{F}$ of a particle with mass m moving with speed u relative to $\mathcal{F}$ is

$$m_r = \gamma(u)m$$

where m is the rest mass of the particle.

Note that the relativistic mass is the same as the rest mass in the inertial rest frame of the particle [should such a frame exist]. This is a rather nice frame-dependant quantity, for we have

Corollary 27

The momentum p and total energy E of a particle with relativistic mass m_r and speed u are

$$p = m_r u, \qquad E = m_r c_0^2.$$

The proper force is

$$F_0 = \frac{d}{dt}(m_r u).$$

Notice in particular that $p = m_r u$ has the exact same form as the Newtonian form of momentum, and, of course, $m_r \sim m$ in the Newtonian limit $u \ll c_0$ since $\gamma(u) \to 1$ as $u \to 0$. We will *always* use a subscript 'r' when denoting a relativistic mass. In addition, when we speak of 'mass', we will mean the rest mass, and not the relativistic mass. If we wish to speak about the relativistic mass, thus, we will always say so explicitly.

3.4.9 Compatibility of Conservation Laws with the Lorentz transformation

We can now prove the promised – and very important – result that the relativistic law of momentum conservation is compatible with the Lorentz transformation. As remarked in section XX above, in order to prove this, we need the additional hypothesis that also the total relativistic *energy* is conserved in the original frame. In addition, it turns out that, as a bonus, we will also be able to prove that the relativistic law of energy conservation is compatible with the Lorentz transformation. Now, let's turn to the proof.

Theorem NN

The laws of conservation of (relativistic) momentum and (relativistic) total energy are compatible with the Lorentz transformation.

Proof[37]

Assume that the total relativistic momentum $\mathbf{p} \coloneqq \sum_{i \in X} \gamma(u_i) m_i \mathbf{u}_i$ of some system is conserved in one inertial frame $\mathcal{F}1 \in \wp\mathcal{F}1$, and let $\mathcal{F}2 \in \wp\mathcal{F}2$ be any other inertial frame in standard configuration relative to $\mathcal{F}1$ with relative speed $v > 0$, in which the total momentum is $\mathbf{p}' \coloneqq \sum_{i \in X} \gamma(u_i') m_i \mathbf{u}_i'$. Now let an arbitrary amount of time pass, after which the momenta are $\tilde{\mathbf{p}} \coloneqq \sum_{i \in X} \gamma(\tilde{u}_i) m_i \tilde{\mathbf{u}}_i$ and $\tilde{\mathbf{p}}' \coloneqq \sum_{i \in X} \gamma(\tilde{u}_i') m_i \tilde{\mathbf{u}}_i'$ in $\mathcal{F}1$ and $\mathcal{F}2$, respectively. Our assumption is thus

$$\mathbf{p} = \tilde{\mathbf{p}}.$$

We will consider one of the particles, the ith, say, but for the sake of notational simplicity, we will drop the subscript 'i'. Let the velocity of the particle be $\mathbf{u} = (u_x, u_y, u_z)$ in $\mathcal{F}1$. In $\mathcal{F}2$ the mass of the particle is still m, since the rest mass is frame-independent. The velocity, however, is

$$\mathbf{u}' = \left(\frac{u_x - v}{1 - u_x v/c_0^2}, \frac{u_y}{\gamma(v)(1 - u_x v/c_0^2)}, \frac{u_z}{\gamma(v)(1 - u_x v/c_0^2)} \right) = \frac{1}{1 - u_x v/c_0^2} \left(u_x - v, \frac{u_y}{\gamma(v)}, \frac{u_z}{\gamma(v)} \right).$$

In $\mathcal{F}1$, the initial and final momenta of *this* particle are

$$\gamma(u) m \mathbf{u} = \gamma(u) m (u_x, u_y, u_z), \quad \text{and}$$

[37] Throughout this proof, we assume that the rest mass of each particle is a constant of motion before, during, and after the collision, since at this point it would not seem natural to assume anything else. However, notice that the proof remains valid even if the pre-collision and post-collision rest masses of each particle are allowed to be different.

$$\gamma(\tilde{u})m\tilde{\mathbf{u}} = \gamma(\tilde{u})m\big(\tilde{u}_x, \tilde{u}_y, \tilde{u}_z\big)$$

(which, of course, might differ) while in $\mathcal{F}2$ the initial momentum is

$$\gamma(u')m\mathbf{u}' = \frac{m}{\left(1-\frac{u_x v}{c_0^2}\right)\sqrt{1-\frac{u'^2}{c_0^2}}}\left(u_x - v, \frac{u_y}{\gamma(v)}, \frac{u_z}{\gamma(v)}\right)$$

where

$$u'^2 \stackrel{\text{def}}{=} |\mathbf{u}'|^2 = \left(\frac{u_x - v}{1-u_x v/c_0^2}\right)^2 + \left(\frac{u_y}{\gamma(v)(1-u_x v/c_0^2)}\right)^2 + \left(\frac{u_z}{\gamma(v)(1-u_x v/c_0^2)}\right)^2$$

so that (this takes a few lines to check)

$$\sqrt{1-\frac{u'^2}{c_0^2}} = \frac{\sqrt{(c_0^2-v^2)(c_0^2-u_x^2-u_v^2-u_z^2)}}{c_0^2-u_x v} = \frac{\sqrt{(c_0^2-v^2)(c_0^2-u^2)}}{c_0^2-u_x v}$$

and (this is immediate, though!)

$$\left(1-\frac{u_x v}{c_0^2}\right)\sqrt{1-\frac{u'^2}{c_0^2}} = \frac{\sqrt{(c_0^2-u^2)(c_0^2-v^2)}}{c_0^2}.$$

Thus the particle's initial momentum (↑) is

$$\begin{aligned}\gamma(u')m\mathbf{u}' &= \frac{mc_0^2}{\sqrt{(c_0^2-u^2)(c_0^2-v^2)}}\left(u_x - v, \frac{u_y}{\gamma(v)}, \frac{u_z}{\gamma(v)}\right) = \\ &= \gamma(u)m\big(\gamma(v)(u_x - v), u_y, u_z\big).\end{aligned}$$

Similarly, the final momentum is found to be

$$\gamma(\tilde{u}')m\tilde{\mathbf{u}}' = \gamma(\tilde{u})m\big(\gamma(v)(\tilde{u}_x - v), \tilde{u}_y, \tilde{u}_z\big).$$

Thus the change of momentum, as seen from $\mathcal{F}2$, is

$$\begin{aligned}\Delta\mathbf{p}' \coloneqq \tilde{\mathbf{p}}' - \mathbf{p}' &= \sum_{i\in X}\gamma(\tilde{u}_i)m_i\big(\gamma(v)\big(\tilde{u}_{i,x} - v\big), \tilde{u}_{i,y}, \tilde{u}_{i,z}\big) - \sum_{i\in X}\gamma(u_i)m_i\big(\gamma(v)\big(u_{i,x} - v\big), u_{i,y}, u_{i,z}\big) = \\ &= \sum_{i\in X}\underline{\mathbf{e}}\begin{pmatrix}\gamma(v)\big[m_i\gamma(\tilde{u}_i)\tilde{u}_{i,x} - m_i\gamma(u_i)u_{i,x} + m_i\gamma(u_i)v - m_i\gamma(\tilde{u}_i)v\big] \\ m_i\gamma(\tilde{u}_i)\tilde{u}_{i,y} - m_i\gamma(u)u_{i,y} \\ m_i\gamma(\tilde{u}_i)\tilde{u}_{i,z} - m_i\gamma(u)u_{i,z}\end{pmatrix} = \\ &= \sum_{i\in X}\underline{\mathbf{e}}\begin{pmatrix}\gamma(v)\big[m_i\gamma(\tilde{u}_i)\tilde{u}_{i,x} - m_i\gamma(u_i)u_{i,x}\big] \\ m_i\gamma(\tilde{u}_i)\tilde{u}_{i,y} - m_i\gamma(u)u_{i,y} \\ m_i\gamma(\tilde{u}_i)\tilde{u}_{i,z} - m_i\gamma(u)u_{i,z}\end{pmatrix} + \sum_{i\in X}(\gamma(v)[m_i\gamma(u_i)v - m_i\gamma(\tilde{u}_i)v])\hat{\mathbf{x}}'.\end{aligned}$$

The first sum is obviously zero by virtue of $\mathbf{p} = \tilde{\mathbf{p}}$. The second sum is zero *if and only if*

$$\begin{aligned}\sum_{i\in X}(\gamma(v)[m_i\gamma(u_i)v - m_i\gamma(\tilde{u}_i)v])\hat{\mathbf{x}}' &= \gamma(v)v\hat{\mathbf{x}}'\sum_{i\in X}\big(m_i\gamma(u_i) - m_i\gamma(\tilde{u}_i)\big) = \\ &= \frac{\gamma(v)v}{c_0^2}\hat{\mathbf{x}}'\sum_{i\in X}(\gamma(u_i)m_i c_0^2 - \gamma(\tilde{u}_i)m_i c_0^2) = \\ &= \frac{\gamma(v)v}{c_0^2}\hat{\mathbf{x}}'\left(\sum_{i\in X}\gamma(u_i)m_i c_0^2 - \sum_{i\in X}\gamma(\tilde{u}_i)m_i c_0^2\right) = -\frac{\gamma(v)v}{c_0^2}\hat{\mathbf{x}}'\cdot\Delta E = 0\end{aligned}$$

that is, *if and only if* we assume that the *change in total energy*

$$\Delta E \coloneqq \sum_{i\in X}\gamma(\tilde{u}_i)m_ic_0^2 - \sum_{i\in X}\gamma(u_i)m_ic_0^2 = 0$$

in $\mathcal{F}1$. We finally show that the total energy is conserved in $\mathcal{F}2$ too. The initial total energy in $\mathcal{F}2$ is

$$E' \coloneqq \sum_{i\in X}\gamma(u_i')m_ic_0^2 = \sum_{i\in X}\frac{m_ic_0^2\left(c_0^2-u_{i,x}v\right)}{\sqrt{\left(c_0^2-v^2\right)\left(c_0^2-u_i^2\right)}}$$

while the final total energy is

$$\tilde{E}' \coloneqq \sum_{i\in X}\gamma(\tilde{u}_i')m_ic_0^2 = \sum_{i\in X}\frac{m_ic_0^2\left(c_0^2-\tilde{u}_{i,x}v\right)}{\sqrt{\left(c_0^2-v^2\right)\left(c_0^2-\tilde{u}_i^2\right)}}.$$

Thus, the change in total energy is

$$\begin{aligned}\Delta E' \coloneqq \tilde{E}' - E' &= \sum_{i\in X}\frac{m_ic_0^2\left(c_0^2-\tilde{u}_{i,x}v\right)}{\sqrt{\left(c_0^2-v^2\right)\left(c_0^2-\tilde{u}_i^2\right)}} - \sum_{i\in X}\frac{m_ic_0^2\left(c_0^2-u_{i,x}v\right)}{\sqrt{\left(c_0^2-v^2\right)\left(c_0^2-u_i^2\right)}} = \\ &= \frac{c_0}{\sqrt{c_0^2-v^2}}\left[\sum_{i\in X}\frac{m_ic_0\left(c_0^2-\tilde{u}_{i,x}v\right)}{\sqrt{c_0^2-\tilde{u}_i^2}} - \sum_{i\in X}\frac{m_ic_0\left(c_0^2-u_{i,x}v\right)}{\sqrt{c_0^2-u_i^2}}\right] = \\ &= \gamma(v)\left[\sum_{i\in X}m_i\gamma(\tilde{u}_i)\left(c_0^2-\tilde{u}_{i,x}v\right) - \sum_{i\in X}m_i\gamma(u_i)\left(c_0^2-u_{i,x}v\right)\right] = \\ &= \gamma(v)\left[\sum_{i\in X}m_i\gamma(\tilde{u}_i)c_0^2 - m_i\gamma(\tilde{u}_i)\tilde{u}_{i,x}v - m_i\gamma(u_i)c_0^2 + m_i\gamma(u_i)u_{i,x}v\right] = \\ &= \gamma(v)\sum_{i\in X}\left[m_i\gamma(\tilde{u}_i)c_0^2 - m_i\gamma(u_i)c_0^2\right] - v\gamma(v)\sum_{i\in X}\left[m_i\gamma(u_i)u_{i,x} - m_i\gamma(\tilde{u}_i)\tilde{u}_{i,x}\right].\end{aligned}$$

The first sum is

$$\sum_{i\in X}\left[m_i\gamma(\tilde{u}_i)c_0^2 - m_i\gamma(u_i)c_0^2\right] \overset{\text{def}}{=} \Delta E = 0$$

and the second is

$$\sum_{i\in X}\left[m_i\gamma(u_i)u_{i,x} - m_i\gamma(\tilde{u}_i)\tilde{u}_{i,x}\right] = \sum_{i\in X}\left[m_i\gamma(u_i)u_i - m_i\gamma(\tilde{u}_i)\tilde{u}_i\right]\cdot\hat{\mathbf{x}} = \Delta\mathbf{p}\cdot\hat{\mathbf{x}} = 0.$$

Therefore, $\Delta E' = 0$ and total energy is also conserved in $\mathcal{F}2$. Thus, we have shown

$$\begin{cases}\Delta\mathbf{p} = 0\\ \Delta E = 0\end{cases} \Rightarrow \begin{cases}\Delta\mathbf{p}' = 0\\ \Delta E' = 0\end{cases}$$

for any primed inertial frame and at any later time, which is a precise statement of the theorem. ■

It is interesting that the concepts of momentum and energy [conservation] are 'entangled' in special relativity. This reminds of the fact that space and time are 'entangled' by the Lorentz transformation. It is an amazing fact that the momentum–energy entanglement is of the exactly same form as the space–time entanglement, as is evident from

Theorem NN

Let $\mathcal{F}1 \in \wp\mathcal{F}1$ and $\mathcal{F}2 \in \wp\mathcal{F}2$ be two inertial frames in standard configuration and with relative speed v. Let a particle have momentum $\mathbf{p} = (p_x, p_y, p_z)$ and energy$E$ as seen from $\mathcal{F}1$, and $\mathbf{p}' = (p'_x, p'_y, p'_z)$ and E' as seen from $\mathcal{F}2$. Then

$$\begin{aligned} E' &= \gamma(v)(E - vp_x) \\ p'_x &= \gamma(v)(p_x - v\,E/c_0^2) \\ p'_y &= p_y \\ p'_z &= p_z \end{aligned}$$

where $\gamma(v) = (1 - v^2/c_0^2)^{-1/2}$ is the Lorentz factor.

Proof

Let the particle have mass m and velocity $\mathbf{u} = (u_x, u_y, u_z)$ in $\mathcal{F}1$. Then it has momentum

$$\begin{aligned} p_x &= \gamma(u)mu_x \\ p_y &= \gamma(u)mu_y \\ p_z &= \gamma(u)mu_z \end{aligned}$$

and total energy

$$E = \gamma(u)mc_0^2.$$

In $\mathcal{F}2$, the rest mass is naturally the same, but the velocity is changed to (↑), transforming the momentum into (↑). But this can be written

$$\begin{aligned} p'_x &= \gamma(u)\gamma(v)m(u_x - v) = \\ &= \gamma(v)(\gamma(u)mu_x - \gamma(u)mv) = \\ &= \gamma(v)(p_x - v\,E/c_0^2) \\ p'_y &= \gamma(u)mu_y \stackrel{\text{def}}{=} p_y \\ p'_z &= \gamma(u)mu_z \stackrel{\text{def}}{=} p_z. \end{aligned}$$

From (↑), we recall that the energy is transformed to

$$E' = \frac{m_i c_0^2(c_0^2 - u_x v)}{\sqrt{(c_0^2 - v^2)(c_0^2 - u^2)}} = m_i\gamma(v)\gamma(u)(c_0^2 - u_x v) = \gamma(v)(\gamma(u)mc_0^2 - \gamma(u)mu_x v) = \\ = \gamma(v)(E - p_x v).$$

■

3.4.10 The Mass–Energy Equivalence

Let us recapitulate on what we have done in SR dynamics so far.

- The law of conservation of Newtonian momentum is incompatible with the postulates of special relativity.[38]
- The law of conservation of relativistic momentum is compatible with the postulates of special relativity. This does not prove that the law is valid; only experimental verification can make us certain of this. Just as experimental verification made Newton postulate the

[38] We have shown that the law is incompatible with the Lorentz transformation, but since the latter follows immediately from the postulates of the theory, the law is actually incompatible with the postulates.

Newtonian law of momentum conservation (via Newton's third law), experimental verification make us certain enough to postulate the relativistic law of momentum conservation. In addition, an important theoretical fact suggesting this law is that it reduces to the Newtonian law (which we are far more familiar with) in the low-speed limit.

In addition, but on a slightly less rigorous, and almost poetical level, we can make a simple observation. Let the phrase 'relativistic generalisation of Newtonian momentum' denote a *new* expression that (1) tends to the expression for the Newtonian momentum in the low-speed limit (2) and is compatible the postulates of special relativity (in particular with the Lorentz transformation). Then we also have the encouraging fact that the relativistic expression $\gamma(u)m\mathbf{u}$ is probably the 'simplest' relativistic generalisation of Newtonian momentum $m\mathbf{u}$ there is.

- We have found a new expression for the kinetic and total energy such that the law of total energy conservation is compatible with the Lorentz transformation. This law also reduces to the Newtonian law in the low-speed limit, and encouraged by experimental verification we postulate this new law. In fact, we *have* to postulate this law if we wish to postulate the law of conservation of momentum, because of the entanglement. That is, either we postulate none of the conversation laws, or we postulate both. Encouraged by both theoretical suggestions (such as the low-speed limits) and experimental verification (especially the latter), we feel little shame in postulating both.

3.4.10.1 An Inelastic Collision: The Rest Mass

We will now consider a simple *inelastic* collision. Let A and B be two billiard balls of equal rest mass m and pre-collision velocities $\mathbf{u}_A = u\hat{\mathbf{x}}$ and $\mathbf{u}_B = -u\hat{\mathbf{x}}$. Assume that their post-collision velocities are $\mathbf{u}_A = -\lambda u\hat{\mathbf{x}}$ and $\mathbf{u}_B = \lambda u\hat{\mathbf{x}}$ for some $\lambda \in [0,1]$. If $\lambda = 1$ the collision is totally elastic, but for all other such λ, the collision is inelastic.

In Newtonian physics, every possible case is well familiar to us. Irrespective of λ, the total momentum of the system A + B is a constant of motion. The simplest case is $\lambda = 1$ in which there isn't really much to say at all, for even the total energy is constant. If $\lambda < 1$ then kinetic (and total mechanical) energy is clearly lost. The lost mechanical energy has been converted into other forms of energy, for instance into thermal energy inside the balls. If the collision took place in vacuum, that should account for the majority of the lost kinetic energy. If the collision takes place on a pool table in a room filled with air, then the surrounding air and the table should get some thermal energy, too. In addition, a pressure (sound) wave would be produced, both inside the table and in the air. In any realistic collision between typical billiard balls, $\lambda \lesssim 1$[39], but if we instead use soft and deformable balls, we can get far lower values of λ, even $\lambda \approx 0$.

We will now investigate the same collision considering the relativistic corrections. The total pre-collision momentum is

$$\mathbf{p} \coloneqq \gamma(u)mu\hat{\mathbf{x}} - \gamma(u)mu\hat{\mathbf{x}} = \mathbf{0}$$

and the total post-collision momentum is

$$\tilde{\mathbf{p}} \coloneqq -\gamma(\lambda u)m\lambda u\hat{\mathbf{x}} + \gamma(\lambda u)m\lambda u\hat{\mathbf{x}} = \mathbf{0}.$$

Thus, momentum is conserved. The total pre-collision energy is

[39] Here, $\lambda \lesssim 1 \Leftrightarrow (\lambda < 1) \wedge (\lambda \approx 1)$.

$$E \coloneqq \gamma(u)mc_0^2 + \gamma(u)mc_0^2 = 2\gamma(u)mc_0^2$$

while the total post-collision energy is

$$\tilde{E} \coloneqq \gamma(\lambda u)mc_0^2 + \gamma(\lambda u)mc_0^2 = 2\gamma(\lambda u)mc_0^2.$$

That is, *total energy is not conserved* if $\lambda < 1$. But according to our postulate of energy conservation, total energy *is* conserved. Hence, we have a contradiction.

One might feel that the problem is due to our postulating the conversation of the total relativistic energy. Indeed, the Newtonian analogue (and limit) of the total relativistic energy $\gamma(u)mc_0^2 = mc_0^2 + E_{k,\mathrm{rel}}$ is the sum of a constant potential and the (Newtonian) kinetic energy, that is, essentially, *kinetic* energy. But we all know that the mechanical (that is, macroscopic) kinetic energy is not a constant of motion in an inelastic collision (or if there are other non-conservative forces present, such as friction). Indeed, there are other forms of energy, such as thermal and electromagnetic energy. That is, from a Newtonian point of view, $E \neq \tilde{E}$ is not a contradiction, but what is to be expected, because we neglect the non-mechanical forms of energy produced in the collision.

Notice, however, that Postulate NN talks strictly about *isolated* systems. That is, if the system exchanges momentum or energy with the surroundings, then the postulate do not apply to the system. In this case, this means that we have to speak only of the case when the billiard balls collide in vacuum, so that no table and no air can steal any energy from the balls. But this still does not resolve our 'paradox', because the system A + B clearly contains the thermal energy inside the balls, and this increases if $\lambda < 1$. Hence, even if the system is perfectly isolated, Postulate NN seems to be violated.

There are basically just two ways to resolve this paradox. One is to abandon Postulate NN, and the other is to let the *rest mass* of each ball change during the collision. From a Newtonian point of view, the latter is absurd, but by now we are used to the fact that special relativity forces us to abandon our old perceptions of physical 'common sense'. It also turns out that the latter approach is the one that agrees with experiments.

Admittedly, our treatment of relativistic energy was not quite as careful as our treatment of relativistic momentum, so the reader might object to the fact that we challenge the Newtonian concept of mass just in order to save the conservation of relativistic total energy. However, if we would accept to invalidate the law of energy conservation, we would also have to reject the law of momentum conservation, because the proof of *compatibility* of the law of momentum conservation requires that the energy be conserved too in the first frame (and vice versa) due to the 'entanglement' of momentum and energy [conservation]. Indeed, insist on the conservation of rest mass, and consider a second frame $\mathcal{F}2$ in standard configuration relative to the frame $\mathcal{F}1$ used in the experiment above. Let the relative speed be $v = u$, so that $\mathcal{F}2$ is the pre-collision rest frame of A. Then, as seen from $\mathcal{F}2$, the x component of the total momentum changes from

$$(p_x')_{\mathrm{before}} = m \cdot \frac{1}{\sqrt{1 - \frac{1}{c_0^2}\left(\frac{u-v}{1-\frac{uv}{c_0^2}}\right)^2}} \cdot \frac{u-v}{1-\frac{uv}{c_0^2}} + m \cdot \frac{1}{\sqrt{1 - \frac{1}{c_0^2}\left(\frac{-u-v}{1+\frac{uv}{c_0^2}}\right)^2}} \cdot \frac{-u-v}{1+\frac{uv}{c_0^2}}$$

and

$$(p_x')_{\text{after}} = m \cdot \frac{1}{\sqrt{1 - \frac{1}{c_0^2}\left(\frac{\lambda u - v}{1 - \frac{\lambda u v}{c_0^2}}\right)^2}} \cdot \frac{\lambda u - v}{1 - \frac{\lambda u v}{c_0^2}} + m \cdot \frac{1}{\sqrt{1 - \frac{1}{c_0^2}\left(\frac{-\lambda u - v}{1 + \frac{\lambda u v}{c_0^2}}\right)^2}} \cdot \frac{-\lambda u - v}{1 + \frac{\lambda u v}{c_0^2}}$$

and it is easy to prove that $(p_x')_{\text{before}} \neq (p_x')_{\text{after}}$.

We therefore accept the fact that the rest mass of a particle, although *invariant* (that is, frame-independent at any given time), is not a constant of motion. Let us return to our initial inelastic collision. Now we have to let m be the pre-collision rest mass of the particles, and introduce $\widetilde{m}$ as the post-collision mass [due to symmetry, the post-collision masses need to be equal].

Hence, the pre-collision total momentum is

$$\mathbf{p} \coloneqq \gamma(u)mu\hat{\mathbf{x}} - \gamma(u)mu\hat{\mathbf{x}} = \mathbf{0}$$

and the total post-collision momentum is

$$\tilde{\mathbf{p}} \coloneqq -\gamma(\lambda u)\widetilde{m}\lambda u\hat{\mathbf{x}} + \gamma(\lambda u)\widetilde{m}\lambda u\hat{\mathbf{x}} = \mathbf{0}.$$

Thus, momentum is necessarily conserved. The total pre-collision energy is

$$E \coloneqq \gamma(u)mc_0^2 + \gamma(u)mc_0^2 = 2\gamma(u)mc_0^2$$

while the total post-collision energy is

$$\tilde{E} \coloneqq \gamma(\lambda u)\widetilde{m}c_0^2 + \gamma(\lambda u)mc_0^2 = 2\gamma(\lambda u)\widetilde{m}c_0^2$$

and our postulate $\tilde{E} = E$ makes sure that the total energy is conserved (which is no longer an impossibility) and some trivial algebra yields

$$\widetilde{m} = \frac{\gamma(u)}{\gamma(\lambda u)}m.$$

Notice in particular that $\lambda = 1 \Rightarrow \gamma(u) = \gamma(\lambda u) \Rightarrow \widetilde{m} = m$. That is, every time I said something like “with foresight, we will avoid inelastic collisions in this discussion” in the precious sections, what I really meant to say was “we assume that the rest mass of every particle is a constant of motion”.

Although total energy is conserved in the collision, the *kinetic* energy is reduced by an amount

$$\Delta E_k \coloneqq \tilde{E}_k - E_k = 2(\gamma(\lambda u) - 1)\widetilde{m}c_0^2 - 2(\gamma(u) - 1)mc_0^2 = -2\Delta mc_0^2$$

where

$$\Delta m \coloneqq \widetilde{m} - m$$

is the change in rest mass of either of the balls. At any time, the total energy is split equal between the two balls (by symmetry), and so we can conclude:

- The total energy as defined by $E = (\gamma(u) - 1)mc_0^2 + mc_0^2 = \gamma(u)mc_0^2$ is conserved.
  - Notice that $\gamma(u)$ is decreased while m is increased. [This is the content of (↑).]
- At the collision, a ball *gains* thermal energy and *gains* rest energy as defined by mc_0^2.
  - Thus, its rest mass m is increased.

- At the collision, a ball *loses* kinetic energy as defined by $(\gamma(u) - 1)mc_0^2$.
- The change in rest energy ΔE_0 due to the change of rest mass Δm is $\Delta E_0 = (\Delta m)c_0^2$.

The above example *suggests* that the mysterious 'rest energy' of an object is a measure of the internal (such as thermal) energy of the object. Since the rest energy $E_0 \stackrel{\text{def}}{=} mc_0^2$ where c_0^2 is a fundamental constant of nature, rest energy is essentially the same thing as mass (but with a different unit). Therefore, if the rest energy is a measure of the internal energy, so is the mass.

Observation NN

The rest energy, and so the rest mass, of a body is a measure of the total internal energy of the body.

We will use this observation as a postulate.

Now that we have constructed a theory in which the mass of a particle does not behave like in the Newtonian theory, it is a very valid question to ask, "what *is* 'mass' in this new theory?" In Newtonian physics, mass is equal to the inertial, active gravitational, and passive gravitational masses, which are fairly well-defined from an experimental point of view. Experimental verification supports the fact, however, that the (rest) mass of particle retains its Newtonian identity. For example, a hot billiard ball is heavier than a cold (otherwise identical) billiard ball, and it is more resistant to a change in velocity. In fact, the title of one of Einstein's famous 1905 papers is "Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?" ("Does the Inertia of a Body Depend Upon Its Energy Content?").

Example NN

Consider the two balls above. Put $m = 1$ kg, $u = 3$ m/s, and $\lambda = 0.9$. Then the pre-collision rest energy of either ball is

$$E_0 = mc_0^2 = 89\,875\,517\,873\,681\,764 \text{ J}$$

while the pre-collision kinetic energy is

$$E_k = (\gamma(u) - 1)mc_0^2 = 4.500\,000\,000\,000 \ldots \text{J}.$$

so that the total energy is

$$E = E_0 + E_k = 89\,875\,517\,873\,681\,768.500 \ldots \text{J}.$$

Clearly the rest mass 'dominates' [to say the least!]. After the collision,

$$\tilde{m} = \frac{\gamma(u)}{\gamma(\lambda u)} m = 1.000000000000000009 51 \ldots \text{kg}.$$

Thus, the mass has increased by

$$\Delta m \coloneqq \tilde{m} - m = 9.51 \cdot 10^{-18} \text{ kg} = 9.51 \text{ fg}.$$

Example NN + 1

Same setup but $\lambda = 0$. The pre-collision energies are the same, that is,

$$\begin{aligned} E_0 &= 89\,875\,517\,873\,681\,764\ \text{J} \\ E_k &= 4.500\,000\,000\,000\ldots\ \text{J} \\ E &= 89\,875\,517\,873\,681\,768.500\ldots\ \text{J}. \end{aligned}$$

According to (↑), the post-collision rest mass is

$$\tilde{m} = \frac{\gamma(u)}{\gamma(\lambda u)} m = \gamma(u) m = 1.000000000000000050 06\ldots\ \text{kg}.$$

with a difference

$$\Delta m \coloneqq \tilde{m} - m = 50.0\ \text{fg}.$$

Hence, the post-collision energies are

$$\begin{aligned} \tilde{E}_0 &= 89\,875\,517\,873\,681\,768.500\ldots\ \text{J} \\ \tilde{E}_k &= 0 \\ \tilde{E} &= 89\,875\,517\,873\,681\,768.500\ldots\ \text{J}. \end{aligned}$$

Obviously, we didn't have to use (↑), since the kinetic energy is zero after the collision, while the total energy is the same as it was prior to the collision. Hence, we get the rest energy, that is, the mass by

$$\tilde{E}_0 = \tilde{m} c_0^2 = \tilde{E} = E.$$

Thus

$$\tilde{m} = E\ c_0^{-2} = \gamma(u) m c_0^2\ c_0^{-2} = \gamma(u) m.$$

3.4.10.2 Some Notes on Additivity

The total relativistic energy, and so the relativistic mass, is an additive property, just as energy and mass are additive in Newtonian mechanics. However, the rest energy, and so the rest mass, is in general *not* additive, and you already know that.

Consider a space colony built inside a huge spherical shell. Assume that there are a lot of shuttles flying around inside it. There might also be trains running on the inside surface of the shell, but no automobile cars (why?). The rest energy (mass) of the colony is the energy (mass) of the colony as measured in a frame of reference, relative to which the colony as a whole (that is, the shell) is stationary. Had the colony been empty, this would only have been due to the shell alone, but now the shuttles and trains inside it will contribute to its rest energy (mass). They will do so using their rest energies (masses), of course, but since the rest energy (mass) of the colony is a measure of its internal energy, it clearly has to include the *kinetic* energy of the shuttles and trains as well (relative to the shell). Thus, the rest mass of the colony is *greater than* the sum of the rest masses of the empty shell, shuttles, and cars, individually.

Similarly, a container of gas has a rest mass greater than the sum of the rest masses of the empty container and all the constituent molecules, since the molecules are not at rest relative to the container of the gas, and so has kinetic energy relative to the container, and thus are contributing to the internal energy of it.

I said you already knew this, and you did. When considering the inelastic billiard-ball collisions above, we concluded that the rest energy (mass) of each ball increased at the collision, due to the increase in thermal energy. But thermal energy is 'simply'[40] kinetic (and potential) energy of the constituent particles of the body. Hence, if you compare the ball's constituent particles with the shuttles (and trains) of the colony's shell, then they are moving slowly before the collision, and are moving rapidly afterwards.

3.4.10.3 Other Forms of Energy

Thus far we have concluded that $E = mc_0^2$ for a stationary body and $E = m_r c_0^2$ for any (stationary or non-stationary) body, where E is the total energy and m the rest mass (m_r the relativistic mass) of the body. This is a restrictive form, or a special case, of the mass–energy equivalence. The full form of the equivalence, that we will simply postulate, states that $E = m_r c_0^2$ applies to *every* system of total energy E or total relativistic mass m_r. For instance, it applies to the electromagnetic field in empty space. Hence, this field has mass!

The Mass—Energy Equivalence

Let a region in space have total relativistic mass m_r. Then the total energy in the region is $E = m_r c_0^2$. Conversely, if a region in space has total relativistic energy E, then the total relativistic mass in the region is $m_r = c_0^{-2} E$.

As an example, consider a box the inside walls of which are ideal mirrors. If there is light in the box, the box is heavier than it would be if there were not as much light in the box.

Notice that we already have concluded the mass–energy equivalence in the case where the system contains only of material bodies. We can now shed some additional light on the relativistic mass. In fact, the relativistic mass of a body is equal to the rest mass *plus* the mass *associated*[41] with the kinetic energy of the body. Indeed, this statement is

$$m_r = m + E_k\, c^{-2} = m + (\gamma(u) - 1)mc_0^2 c_0^{-2} = m + \gamma(u)m - m = \gamma(u)m$$

which is true by definition.

Combining Postulate NN with the Mass–Energy Equivalence, we have

Corollary NN

The total relativistic mass of an isolated system is a fundamentally conserved quantity.

Indeed, if mass and energy is the same thing (up to the unit), then conservation of either one implies conservation of the other one.

[40] We neglect all quantum effects.

[41] If E is *some* quantity with the dimension of energy, and m is *some* quantity with the dimension of mass, we say that the quantities are *associated* /with one another/ iff $E = mc_0^2$. Thus, it is always true, by definition, that the rest (or relativistic) mass of an object is associated with the rest (or total) energy of the same object.

As an example, consider the annihilation of an electron and a positron. In the centre of mass frame, they both approach each other with equal speed. Before the annihilation, they have both relativistic mass and total energy, related by $E = m_r c_0^2$. Since they are in motion, the total energy is equal to the rest energy plus the non-zero kinetic energy, and the relativistic mass is greater than the rest mass. When they meet, they both disappear and two photons are created. These are receding from each other. Indeed, since the momentum was zero before the annihilation, it has to be zero afterwards as well. [Compton scattering shows that photons do have momentum.] The total relativistic energy is conserved, so the frequency of the photons is determined. But the relativistic mass is also conserved: the photons have energy, and therefore they have relativistic mass. In fact, the total relativistic mass of the two photons equals the total relativistic mass of the two leptons.

A common misconception is that $E = mc_0^2$ states that mass can be *converted* into energy and vice versa. This is wrong. Both (total relativistic) energy and (relativistic) mass are conserved; in fact, they are essentially the same thing. When the electron and positron annihilate, mass is not destroyed and energy created. The energy of the photons did already exist in the energy (rest + kinetic) of the leptons, and the (relativistic) mass of the leptons is later on associated with the photons. Moreover, this is not only a way of seeing things; recall that a box full of photons *is* heavier (as shown by a hypothetical, unreasonably high-accuracy laboratory scale) than a dark box.

However, although (relativistic) *mass* cannot be destroyed or created, *matter* can, if you define the term properly. For instance, if you consider leptons to be matter, but not photons, then matter was clearly destroyed in the annihilation. Conversely, in pair production, matter is created.

The conversion factor c_0^2 in $E = mc_0^2$ if a huge number, namely,

$$c_0^2 = 89875517873681764\,\mathrm{m}^2/\mathrm{s}^2.$$

If one gram of *matter* was converted to some useful form of energy (such as electrical energy), we would thus obtain an amount

$$E = mc_0^2 = 90\ \mathrm{TJ}$$

of useful energy. If you loaded this amount of energy into a battery or capacitor of some sort, it would become one gram heavier.

As a final example, let one body emit a beam of photons, which are absorbed by another body. If the total energy E is transmitted, that is, *moved* from the first body to the latter one, then the rest mass of the first body will decrease by an amount Ec_0^{-2} while the rest mass of the second will increase by the same amount.[42] During the process, a volume in space between the bodies that is containing photons with a total energy of ΔE will have relativistic mass $\Delta E c_0^{-2}$. Einstein himself wrote in his 1905 paper (Einstein, 1905),

> *Wenn die Theorie den Tatsachen entspricht, so iibertragt die Strahlung Trtigheit zwischen den emittierenden und absorbierenden Korpern.*
>
> (If the theory corresponds to the facts, radiation conveys inertia between the emitting and absorbing bodies.)

[42] Notice that, if you only consider one of the objects in its rest frame, the first one, say, then its total energy (its rest mass=its relativistic mass) will decrease. But this does not contradict Corollary NN, of course, because if the system emits radiation to the surroundings, then clearly it is not isolated.

3.4.11 The Energy Triangle

Consider a matter particle of mass m and velocity $\mathbf{u}$. Then the momentum is

$$\mathbf{p} = \gamma(u)m\mathbf{u}$$

and the energy is

$$E = \gamma(u)mc_0^2.$$

Therefore

$$(pc)^2 + (mc_0^2)^2 = \gamma(u)^2 m^2 u^2 c^2 + m^2 c_0^4 = \frac{m^2 u^2 c^2}{1 - \frac{u^2}{c^2}} + m^2 c_0^4 = \frac{m^2 u^2 c^4}{c^2 - u^2} + m^2 c_0^4 =$$
$$= \left(\frac{u^2}{c^2 - u^2} + 1\right) m^2 c_0^4 = \left(\frac{c^2}{c^2 - u^2}\right) m^2 c_0^4 = \gamma(u)^2 m^2 c_0^4 = E^2$$

and we have

Proposition NN

Let a matter particle have rest mass m and velocity $\mathbf{u}$. Let E and p be its total energy and momentum, respectively. Then

$$E^2 = (mc_0^2)^2 + (pc_0)^2.$$

Notice that the first term is due to the rest energy and the second is due to the kinetic energy. [But $E_k \neq pc_0$ because $E^2 = (E_0 + E_k)^2 \neq E_0^2 + E_k^2$.]. If the particle is at rest relative to the observer, $p = 0$ and so we recover Einstein's $E = mc_0^2$.

In the spirit of relativity theory, we now generalise this to apply to *any* particle, including 'massless' particles such as photons. [Thus, we postulate it.] We need to explain why we put quotation marks around 'mass-less'. Classically, a photon is said to be massless, and even in relativity theory, we say that the photon has zero rest mass. But a photon has energy, and so, by the Mass–Energy equivalence, it has *relativistic* mass. But how is this possible? Indeed, $m = 0 \Rightarrow m_r = \gamma(u)m = 0 \Rightarrow E = 0$? The 'explanation' is that a photon travels with the speed of light. Hence $\gamma(u)$ is not defined. However, it might seem 'plausible' that $m_r = m\gamma(u)$ is finite if m is 'infinitely small' and $\gamma(u)$ is 'infinitely big'. Anyhow, we *postulate* that a photon travels with the speed of light (this is the second of Einstein's postulates), has zero rest mass and finite energy (=relativistic mass). Also, we postulate (↑) for photons, too, which in the case of $m = 0$ reads

$$E = pc_0.$$

Notice that the mass–energy equivalence dictates

$$E = m_r c_0^2$$

and so

$$p = Ec_0^{-1} = m_r c_0$$

which is the same relation between momentum, relativistic mass, and speed, as holds for material particles. We end this section by remarking that

$$\mathbb{R} \ni E = \gamma(u)mc_0^2$$

by our non-stringent discussion above implies

Observation NN

A particle with finite (and non-zero) energy has zero rest mass if and only if it travels with the speed of light.

By the kinematics of special relativity, if a speed is found to be equal to the speed of light in *one* frame, then the speed is equal to the speed of light in *any* frame. In particular, a photon has the speed of light in *any* frame, and so Observation NN tells us that the rest mass of a photon is the same in any frame, which is consistent with the frame-independent nature of the rest mass as known from the dynamics of matter particles.

By the way, the title of this subsection is the name of a simple and self-explanatory mnemonic:

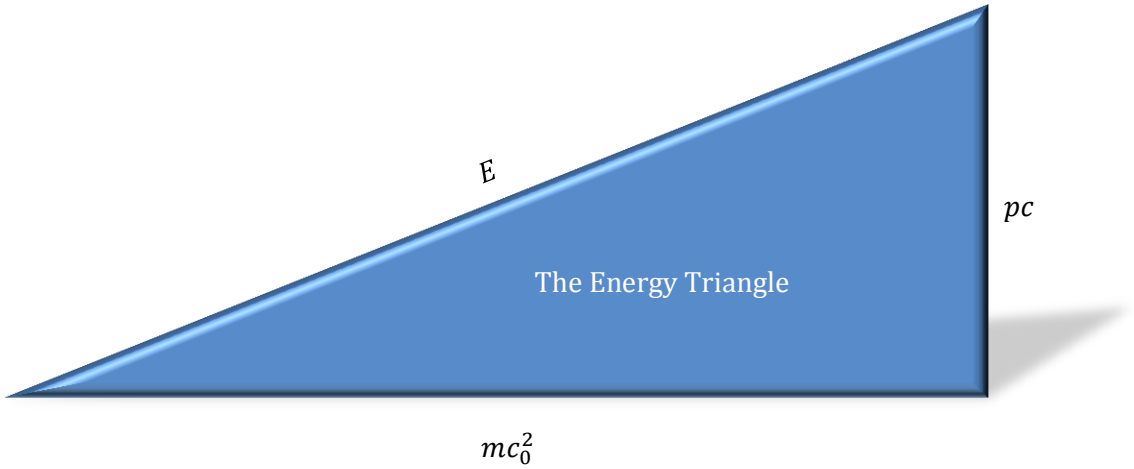


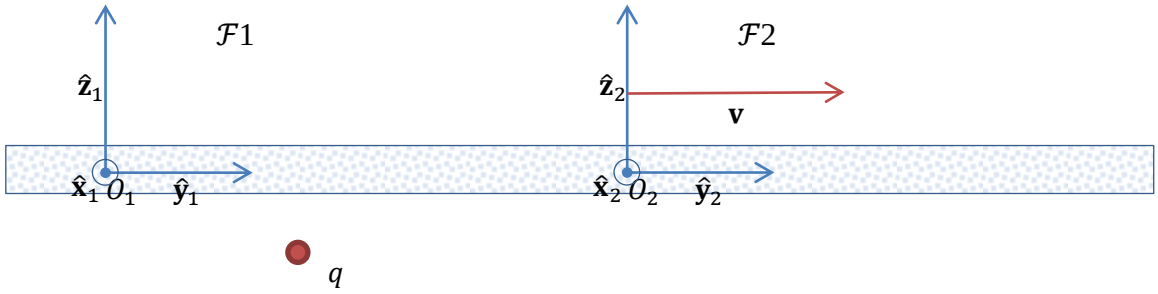
Figure 31. The Energy Triangle.

3.4.12 Summary

This long section has been concerned about the dynamics of special relativity. We have developed the standard theory for special relativity, and I have tried to do it in such a careful way as possible. We have shown which laws of conservation are *compatible* with Einstein's postulates, and then we have postulated these laws. We have then seen that at least a weak form of the mass–energy result follows from these postulates, and then we generalised this result in a way that seemed 'natural'. The dynamical theory thus obtain has to be tested against experimental observation. We have *suggested* a dynamical theory, many major results of which have not been entirely proven from more fundamental postulates, and therefore, only experimental verification can make us certain of the validity of the theory. Fortunately, the special theory of relativity is used in every-day physical experiments and consumer electronics, and so we are fairly confident on its validity.

3.5 Relativistic Electrodynamics

We will now continue the discussion on electrodynamics. Recall that Maxwell's equations are not invariant under a Galilean transformation, and that this was a major theoretical problem in Newtonian mechanics, and, therefore, a motivation for special relativity. We will now show that the Lorentz transformation comes to rescue. Consider the same experimental setup as we used when discussing the Galilean transformation, that is, Figure 27. For convenience, we repeat it here:



We will perform the same kind of analysis as we did in the Newtonian case, but we will now include relativistic effects, that is, we will use the Lorentz transformation instead of the Galilean transformation when switching between $\mathcal{F}1$ and $\mathcal{F}2$. The situation in $\mathcal{F}1$ is the same now as in the Newtonian case; there is only the electrostatic field

$$\mathbf{E} = -\frac{\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_1$$

at the test particle. Thus, the Lorentz force law yields the force

$$\mathbf{F} = -\frac{q\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_1$$

on the particle. But what does 'force' mean? In this section, we will use the concept of relativistic force (cf. Section 3.4.4). But in this case it doesn't really matter, for $u = 0$ and so $\gamma(u) = 1$. Hence,

$$\mathbf{a} = -\frac{q\rho}{2\pi\epsilon_0 md}\hat{\mathbf{z}}_1.$$

Let us now investigate the setup from the point of view of $\wp\mathcal{F}2$. Length contraction will *increase* the linear charge density [unit C/m] by a factor of $\gamma(v)$. Thus, the observed charge density is

$$\rho' \coloneqq \gamma(v)\rho.$$

The electrostatic field is now

$$\mathbf{E}' = -\frac{\rho'}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_2 = -\frac{\gamma(v)\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_2$$

since the vertical distance d is unaffected by the standard-configuration Lorentz transformation. In $\wp\mathcal{F}2$, the wire is observed to carry a constant current

$$\mathbf{J} = -\rho'\mathbf{v} = -\gamma(v)\rho\mathbf{v}$$

corresponding to a scalar current of

$$I = |\mathbf{J}| = \gamma(v)\rho v$$

and thus producing a non-vanishing magnetic field

$$\mathbf{B}' = \frac{\mu_0 I}{2\pi d}\hat{\mathbf{x}}_2.$$

The Lorentz force is therefore

$$\mathbf{F}' = q\mathbf{E}' + q(-\mathbf{v}) \times \mathbf{B}' = \underbrace{-\frac{q\gamma(v)\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_2}_{\text{electric force}} + \underbrace{\frac{qv\mu_0 I}{2\pi d}\hat{\mathbf{z}}_2}_{\text{magnetic force}} = -\frac{q\rho}{2\gamma(v)\pi\epsilon_0 d}\hat{\mathbf{z}}_2.$$

Using the relation between the relativistic force and the acceleration (↑),

$$m\frac{\gamma(u)^3}{c_0^2}(-\mathbf{v}\cdot\mathbf{a}')(-\mathbf{v}) + m\gamma(u)\mathbf{a}' = -\frac{q\rho}{2\gamma(v)\pi\epsilon_0 d}\hat{\mathbf{z}}_2$$

where $\mathbf{a}'$ is the acceleration relative to $\mathcal{F}2$. Since $\mathbf{v} = (0, v, 0)$

$$\mathbf{v}\cdot\mathbf{a}' = va_y'$$

and so

$$m\frac{\gamma(u)^3}{c_0^2}\left(-va_y'\right)(-v\hat{\mathbf{y}}_2) + m\gamma(u)\mathbf{a}' = -\frac{q\rho}{2\gamma(v)\pi\epsilon_0 d}\hat{\mathbf{z}}_2$$

which is solved to yield

$$\mathbf{a}' = -\frac{q\rho}{2\gamma(v)^2\pi\epsilon_0 md}\hat{\mathbf{z}}_2 - \frac{\gamma(u)^2}{c_0^2}va_y'v\hat{\mathbf{y}}_2.$$

In components,

$$\begin{aligned} a_x' &= 0 \\ a_y' &= -\frac{\gamma(u)^2}{c_0^2}va_y'v \\ a_z' &= -\frac{q\rho}{2\gamma(v)^2\pi\epsilon_0 md}; \end{aligned}$$

clearly,

$$a_y' = 0.$$

Thus,

$$\begin{aligned} a_x' &= 0 \\ a_y' &= 0 \\ a_z' &= -\frac{q\rho}{2\gamma(v)^2\pi\epsilon_0 md}. \end{aligned}$$

Now we have found the acceleration of the particle both as seen from $\wp\mathcal{F}1$ and from $\wp\mathcal{F}2$. When we did this within the framework of Newtonian mechanics, we found the result incompatible with the Newtonian transformation of acceleration (the Galilean, which is simply $\mathbf{a}' = \mathbf{a}$). We will now see if our relativistic result is compatible with the relativistic (Lorentz) transformation of acceleration, that is, with Proposition NN.

Transforming (↑) back to $\mathcal{F}1$, we find

$$\begin{aligned} a_x &= 0 \\ a_y &= 0 \end{aligned}$$

$$a_z = \frac{-\dfrac{q\rho}{2\gamma(v)^2\pi\epsilon_0 md}}{\gamma(v)^2\left(1-\dfrac{v^2}{c_0^2}\right)^2} = \frac{-\dfrac{q\rho}{2\gamma(v)^2\pi\epsilon_0 md}}{\gamma(v)^{-2}} = -\frac{q\rho}{2\pi\epsilon_0 md}.$$

Compare this with the expected result (↑) above; they are identical. Thus, we see that special relativity *does* resolve this paradox. Special relativity, and not Newtonian mechanics, seems perfectly compatible with the theory of electromagnetism.

3.5.1 The Rise of Magnetism

In this section, we will show that electric and magnetic forces are in fact *not* two different kinds of forces. Instead, we will find that, in a sense, there are only electric forces, and that the magnetic 'forces' are nothing more than 'relativistic corrections' of them.

Even though we have shown electrodynamics to be incompatible with Newtonian physics, it does not require special relativity to appreciate the fact that an electromagnetic field that *some* observer considers purely electric might be considered purely magnetic, or electric + magnetic, by some *other* observer. In fact, this follows immediately from the Maxwell theory, and we saw it already in that chapter. However, now we will be able to show this *without* the use of Maxwell's theory.

More precisely, we will assume that there are only electric forces, that is, we forget about everything related to magnetism. The electric forces are described by Coulomb's law or the vector-analytical form of it, namely, Gauss' law (the Maxwell equation):

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}.$$

We will then show that there cannot exist any electric force at all, unless there exists also some new kind of force, which we will identify with the magnetic force. We will take a slight detour, however: To point out that magnetism is really a *relativistic* correction, we will first do the analysis using mere Newtonian mechanics. We will see that, within the Newtonian theory, electric forces *can* indeed live by themselves.

Consider so once again an infinitely long wire along the y axis. Let its linear charge density [C/m] be ρ, and let it carry no current relative to $\mathcal{F}1 \in \wp\mathcal{F}1$. A single charge (initially) at rest a distance d below the wire will thus experience the electric force

$$\mathbf{F} = -\frac{q\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_1$$

giving it an acceleration (recall that we are using the Newtonian concept of force)

$$\mathbf{a} = \frac{1}{m}\mathbf{F} = -\frac{q\rho}{2\pi\epsilon_0 md}\hat{\mathbf{z}}_1.$$

Now consider again the frame $\mathcal{F}2 \in \wp\mathcal{F}2$, moving with velocity $\mathbf{v} = v\hat{\mathbf{y}}$ relative to $\mathcal{F}1$, just as before. In this frame (recall that there is no length contraction in the Newtonian theory, and that we are neglecting the magnetic force!), the force is still

$$\mathbf{F}' = -\frac{q\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_2.$$

Needless to say, the acceleration $\mathbf{a}' = (1/m)\mathbf{F}'$ is also the same. Thus, there is no problem with a lone electric force [in *this* case, at least]. Now, let us do this 'for real', considering the full set of

relativistic effects. In $\mathcal{F}1$, the expression for the electric force is still the same. The acceleration is now given by the relativistic force equation (↑), but, as we saw earlier, since $\gamma(u) = 1$, no relativistic effects reveal themselves, so still

$$\mathbf{a} = -\frac{q\rho}{2\pi\epsilon_0 md}\hat{\mathbf{z}}_1.$$

From the point of view of $\wp\mathcal{F}2$, however, the linear charge density is $\rho' = \gamma(v)\rho$ so that the electric force

$$\mathbf{F}' = -\frac{q\gamma(v)\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_2.$$

This (we think) is also the total force, for we are not aware of anything called 'magnetism'. However, since the particle is now moving relative to the frame, we need to be careful when using the relativistic force equation. This is literally

$$m\frac{\gamma(v)^3}{c_0^2}(-\mathbf{v}\cdot\mathbf{a}')(-\mathbf{v}) + m\gamma(v)\mathbf{a}' = -\frac{q\gamma(v)\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_2.$$

Working out the components as we did in the last section, we end up with

$$\begin{aligned} a_x' &= 0 \\ a_y' &= 0 \\ a_z' &= -\frac{q\rho}{2\pi\epsilon_0 md}. \end{aligned}$$

This is the (instantaneous) acceleration of the particle relative to $\mathcal{F}2$. But considering the acceleration relative to $\mathcal{F}1$ and applying Proposition NN, we know that the acceleration relative to $\mathcal{F}2$ *has* to be

$$\begin{aligned} a_x &= 0 \\ a_y &= 0 \\ a_z &= -\frac{q\rho}{2\gamma(v)^2\pi\epsilon_0 md} \end{aligned}$$

which clearly contradicts (↑). Thus, when full relativistic effects are considered, we see that there cannot exist electric forces, unless (for instance) they are also accompanied by some 'new' type of force. Apparently (as we saw in the last section), the force of magnetism fits just perfectly, but perhaps there are other possibilities? To rule them out, we will continue our discussion by *deriving* the expression for the magnetic force (in this case).

The acceleration relative to $\mathcal{F}2$ has to be

$$\mathbf{a}' = -\frac{q\rho}{2\gamma(v)^2\pi\epsilon_0 md}\hat{\mathbf{z}}_2.$$

Using the relativistic force equation (↑), this implies that the relativistic force on the particle is necessarily

$$\begin{aligned}\mathbf{F}' &= m\frac{\gamma(u)^3}{c_0^2}(\mathbf{u}\cdot\mathbf{a}')\mathbf{u} + m\gamma(u)\mathbf{a}' = \\ &= m\frac{\gamma(v)^3}{c_0^2}\left(-va_y'\right)(-v\hat{\mathbf{y}}_2) + m\gamma(v)\mathbf{a}' = \\ &= m\frac{\gamma(v)^3}{c_0^2}(-v\cdot 0)(-v\hat{\mathbf{y}}_2) + m\gamma(v)\left(-\frac{q\rho}{2\gamma(v)^2\pi\epsilon_0 md}\hat{\mathbf{z}}_2\right) = \\ &= -\frac{q\rho}{2\gamma(v)\pi\epsilon_0 d}\hat{\mathbf{z}}_2.\end{aligned}$$

Of course, we can write this as

$$\mathbf{F}' = -\frac{q\rho}{2\gamma(v)\pi\epsilon_0 d}\hat{\mathbf{z}}_2 = -\frac{q\rho}{2\pi\epsilon_0 d}\left(\frac{1}{\gamma(v)}\right)\hat{\mathbf{z}}_2 = -\frac{q\rho}{2\pi\epsilon_0 d}\left(\gamma(v) + \left(\frac{1}{\gamma(v)} - \gamma(v)\right)\right)\hat{\mathbf{z}}_2) =$$
$$= -\frac{q\gamma(v)\rho}{2\pi\epsilon_0 d}\hat{\mathbf{z}}_2 + \frac{v^2 q\gamma(v)\rho}{2\pi c_0^2\epsilon_0 d}\hat{\mathbf{z}}_2 =: \mathbf{F}_{\text{electric}} + \mathbf{F}_{\text{unknown}}.$$

The first term we recognize as the electric force, but we are 'astonished' to find a second term. But, if we define

$$\mu_0 \coloneqq \frac{1}{c_0^2\epsilon_0}$$

then the second term reads

$$\mathbf{F}_{\text{unknown}} = \frac{\mu_0 v^2 q\gamma(v)\rho}{2\pi d}\hat{\mathbf{z}}_2 = \frac{qv\mu_0 I}{2\pi d}\hat{\mathbf{z}}_2$$

which is precisely the magnetic force on the charge, as seen in (↑), where the constant μ_0 is also equal to $1/c_0^2\epsilon_0$.

With some ingenuity and effort, the reader can probably himself construct further examples of electric systems in which special relativity requires the electric forces to be accompanied by magnetic forces that agree with the Maxwell theory.

You could *possibly* say that the force of magnetism, which was known *empirically* during the 19th century, has now been *derived* using the special theory of relativity. However, this is not entirely the case, since the theory of relativity is based on Einstein's postulates, the first of which (the constancy of the speed of light) was to a big extent suggested by the Maxwell theory of electromagnetism. Nevertheless, it is not *unimaginable* that one can appreciate the axioms of special relativity without prior knowledge of magnetism.

3.6 Four-Vector Formulation and Spacetime Geometry

In the previous sections, we have in great detail investigated the physical basis of special relativity, and the transition from Newtonian physics to special relativity. Although transparent, the investigation was at times messy and gave rise to rather awkward formulae. We also had to work a lot with the issue of *compatibility*; for instance, we spent quite some time investigating whether or not the relativistic law of momentum conservation is compatible with the axioms of special relativity. This we had to do, for the result is nontrivial, and, in addition, should we have found the law *not* to be compatible with the axioms, then we would have had to abandon it altogether.

Put differently, we were given an equation

$$\mathbf{p}_1 = \mathbf{p}_2$$

between two vectors relative to some inertial frame $\mathcal{F}1 \in \wp\mathcal{F}1$. In this case, the vectors are momenta. Since the momentum is a frame-dependent quantity, there is no *a priori* reason to believe that the vector equation

$$\mathbf{p}_1' = \mathbf{p}_2'$$

should hold in some *other* frame $\mathcal{F}2 \in \wp\mathcal{F}2$, where $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are the corresponding momenta relative to $\mathcal{F}2$. It took quite some effort to show this, as we did in Theorem NN.

In this chapter, we will reformulate the special theory of relativity using spacetime and *four-vectors*, a new kind of mathematical object. Just as we describe nature using numbers and three-component spatial vectors in the Newtonian theory, in special relativity, we describe nature using numbers and four-vectors. This new formulation will generally be more succinct, and – most importantly – will resolve the problem of compatibility entirely. Indeed, if two four-vectors are equal in *some* inertial frame, then they will be equal in *any other* inertial frame connected to the first frame by a Lorentz transformation.

3.6.1 Spacetime

The entanglement of space and time suggests that we should treat space and time not as two separate objects, but rather as a single entity, which we will call *spacetime*. By definition, spacetime M is the set (with the structure of a differentiable manifold – we'll get there) of all possible *events*, an event being a 'place' in spacetime where a particle can exist. In the general theory of relativity, we will pursue this idea very far, but in both Newtonian physics and special relativity, you can introduce a coordinate system in M such that an event is a pair $(t, \mathbf{x})$ of a time coordinate t (in the physical sense) and three spatial coordinates $\mathbf{x} = (x, y, z)$. In other words,

$$M = \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$$

is a real vector space, but the remarks of Section **Fel! Hittar inte referenskälla.** still apply. We eed a coordinate system in spacetime. We will therefore assume that we have settled for some inertial frame $\mathcal{F}1$ of reference. In particular, this means that have chosen to identify some point in spacetime with the origin $(0, 0, 0, 0)$ of $\mathbb{R}^4$. We have also chosen some unit of time to correspond to a unit change in the first (time) coordinate, and we have chosen three geometric spatial vectors to correspond to the directions $\hat{\mathbf{x}} = (0, 1, 0, 0)$, $\hat{\mathbf{y}} = (0, 0, 1, 0)$, and $\hat{\mathbf{z}} = (0, 0, 0, 1)$ of $\mathbb{R}^4$.

Of course, since we consider spacetime as a four-dimensional space (essentially $\mathbb{R}^4$), we could introduce a 'pathological' basis that intermix the temporal coordinate with the special coordinates. This is an obvious mathematical idea and even the case in Newtonian 'spacetime'. Indeed,

assume pure Newtonian physics, and consider spacetime $M = \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$. Choose to identify the direction $\hat{\mathbf{t}} = (1,0,0,0)$ with the direction of time, and let $\hat{\mathbf{x}} = (0,1,0,0)$, $\hat{\mathbf{y}} = (0,0,1,0)$, and $\hat{\mathbf{z}} = (0,0,0,1)$ be an orthogonal spatial basis in the Newtonian sense. Thus, at each time t, space is the subset

$$\Sigma_t = \{t\} \times \text{span}(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}).$$

Let $\underline{\mathbf{e}} = (\hat{\mathbf{t}} \quad \hat{\mathbf{x}} \quad \hat{\mathbf{y}} \quad \hat{\mathbf{z}})$ be the full spacetime basis. This is a 'normal', or 'non-pathological', basis. Then define a new basis $\underline{\mathbf{f}} = (\hat{\mathbf{t}}' \quad \hat{\mathbf{x}}' \quad \hat{\mathbf{y}}' \quad \hat{\mathbf{z}}')$ by

$$\underline{\mathbf{f}} = \underline{\mathbf{e}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

this is also a basis for $\mathbb{R}^4$. However, now the new first coordinate, t', does *not* correspond to the physical concept of time, and the new basis vectors $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$, and $\hat{\mathbf{z}}'$ does *not* form a Newtonian spatial basis. For example, consider the two events $(0,1,2,3)$ and $(0,2,1,2)$ relative to the 'normal' basis. These occur at the same time [and time is absolute in Newtonian physics]. But their coordinates relative to our new and 'pathological' basis are $(-1,1,2,3)$ and $(-2,2,1,2)$, respectively. Thus, of the four coordinates in our new coordinate system, clearly none represents time by itself. From now on, and for the remainder of the chapter on special relativity, we will only be using coordinate systems in spacetime such that the *first* coordinate, by itself, represents time, and the remaining three coordinates are purely spatial. Such a coordinate system, if the frame is inertial, and the spatial coordinates are Cartesian, will be called a *Minkowski system* (or *frame*).

3.6.1.1 The Minkowski 'Inner Product'

We will now introduce a function $\langle \cdot, \cdot \rangle : M \times M \to \mathbb{R}$ by

$$\langle u, v \rangle \coloneqq \langle (u_t, u_x, u_y, u_z), (v_t, v_x, v_y, v_z) \rangle \coloneqq u_t v_t - u_x v_x - u_y v_y - u_z v_z.$$

Clearly, $\langle \cdot, \cdot \rangle$ is *not* an inner product on $\mathbb{R}^4$ in the usual sense, because $\langle u, u \rangle$ might be negative, and might be zero even if $u \neq 0$. Nevertheless, we will still call it an inner product. This is standard, and very convenient, in relativity theory. We will also use this inner product to form a norm-square in the usual way. However, one must remember that all of these objects are defined using the relaxed condition on the inner product. In particular, this means that the norm-square of a vector might be negative, and a non-zero vector might be of zero norm.

Definition NN

On spacetime, we introduce the inner product

$$\langle u, v \rangle \coloneqq \langle (u_0, u_1, u_2, u_3), (v_0, v_1, v_2, v_3) \rangle \coloneqq u_0 v_0 - u_1 v_1 - u_2 v_2 - u_3 v_3$$

and the norm-square

$$\|u\|^2 \coloneqq \langle u, u \rangle.$$

Notice that this inner product is *not* an inner product in the usual sense, since it violates the requirements $\langle u, u \rangle \geq 0 \; \forall u \in M$ and $\langle u, u \rangle = 0 \Leftrightarrow u = (0,0,0,0)$.

The inner-product space $(M, \langle \cdot, \cdot \rangle)$ is known as *Minkowski space* or *Minkowski spacetime*. If $\langle u, v \rangle = 0$, then u and v are said to be *orthogonal*. If $\|u\|^2 = c_0^2$, then u is a *unit vector*.

We make

Definition NN

Let

$$\eta \coloneqq \operatorname{diag}(1,-1,-1,-1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

be the *Minkowski metric*.[43]

as to obtain

Proposition NN

Let u and $v \in M$. Then

$$\langle u, v \rangle = v^T \eta u = \begin{pmatrix} v_0 & v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

3.6.1.2 The Worldline of a Particle

Consider a particle. During its existence, it will trace out a curve in spacetime. This curve, which is called the *worldline* Γ_W of the particle, is precisely defined by

$$\Gamma_W = \{(t,x,y,z) \in M: \text{ the particle is located at } (x,y,z) \text{ at time } t\}.$$

The worldline can clearly be parameterised with the coordinate time t, that is, there exists functions $t \mapsto x(t)$, $t \mapsto y(t)$, and $t \mapsto z(z)$ such that the particle is located at $\big(t, x(t), y(t), z(t)\big)$ at coordinate time t. However, every coordinate time t corresponds to a proper time τ, as recorded by a clock attached to the particle. It will turn out to be convenient to use the proper time as the parameter, and not the coordinate time. (It is not important what the origin of the proper time is.)

3.6.2 The Lorentz transformation, Four-Vector, and Lorentz Scalars

We first make

Definition NN

Assume that (t,x,y,z) are the coordinates of some event relative to a Minkowski frame $\mathcal{F}1$. Then the *four-coordinates* of the event are $(c_0 t, x, y, z)$.

There is an obvious bijection between the set of possible coordinates and the set of possible four-coordinates. Hence, in principle, they are the same. However, it will turn out that the four-coordinates are neater to work with formally compared with the ordinary coordinates, so from now on we will mainly use this new concept.

[43] The word 'metric' will be explained in the chapter on geometry. Until then, you might call η the 'Minkowski matrix'.

Now, let $\mathcal{F}1 \in \wp\mathcal{F}1$ and $\mathcal{F}2 \in \wp\mathcal{F}2$ be two inertial frames in standard configuration with relative speed v. Assume that a particle has four-coordinates (c_0t, x, y, z) relative to $\mathcal{F}1$, and four-coordinates (c_0t', x', y', z') relative to $\mathcal{F}2$. Then

$$\begin{pmatrix} c_0t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(v) & -\gamma(v)v/c_0 & 0 & 0 \\ -\gamma(v)v/c_0 & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0t \\ x \\ y \\ z \end{pmatrix}$$

according to the Lorentz transformation (Theorem NN).

Definition NN

The Lorentz transformation $\Lambda: M \to M$ has the matrix

$$\Lambda = \begin{pmatrix} \gamma(v) & -\gamma(v)v/c_0 & 0 & 0 \\ -\gamma(v)v/c_0 & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that the Lorentz transformation Λ is an endomorphism on spacetime, given by a symmetric matrix.[44] Recall that Λ connects two inertial frames in standard configuration. Hence, it is *not* the case that Λ connects an *arbitrary* pair of inertial frames. Although when we speak of '*the* Lorentz transformation' we refer to Λ as given above, in the most general sense of the word, a Lorentz transformation is, by definition, a function connecting any pair of inertial reference frames *with the same origin in spacetime*. The set of all Lorentz transformations form a group under composition, called the Lorentz group. The set of transformations connecting *any* pair of inertial reference frames also forms a group, called the Poincaré group. Clearly, the Lorentz group is a proper subgroup of the Poincaré group.

From a physical point of view, it should be enough to investigate the theory by only considering the standard-configuration Lorentz transformation Λ. Indeed, any transformation in the Poincaré group can be written as a composition of spacetime translations, *spatial* rotations, and Λ, and of these three types of transformations, only the last one should be of any non-trivial physical significance. For simplicity, we will therefore restrict our analysis to the case of frames connected via Λ.

Definition NN

Let $X \in \mathbb{R}^4$ be a 4-tuple of numbers relative to an inertial frame $\mathcal{F}1$, and let $X' \in \mathbb{R}^4$ be the corresponding 4-tuple relative to any other inertial frame $\mathcal{F}2$ in standard configuration with $\mathcal{F}1$. If

$$X' = \Lambda X$$

where Λ is the Lorentz transformation between $\mathcal{F}1$ and $\mathcal{F}2$, then X is called a *four-vector*, and is written $\vec{X}$.

[44] We can already now appreciate the reason why the four-coordinates are more 'natural' than the usual coordinates. For one thing, every component of the four-coordinates has the same unit, namely, the metre. In addition, it is clear from Theorem NN that the 'Lorenz transformation' between the usual coordinates is *not* given by a symmetric matrix.

That is, the arrow above a 4-tuple reminds us that the 4-tuple transforms according to (↑) between two inertial frames in standard configuration. We will often denote a four-vector

$$\vec{a} = (a_0, \mathbf{a}) \coloneqq (a_0, a_1, a_2, a_3)$$

where $\mathbf{a}$ is an ordinary spatial vector in $\mathbb{R}^3$, that is, $\mathbf{a} = (a_1, a_2, a_3)$. We will also call $\mathbf{a}$ the 'spatial part' of $\vec{a}$. The reason why this is convenient will reveal itself in just a few lines of text. From Definition NN we immediately have

Corollary NN

The four-coordinates

$$\vec{x} = (c_0 t, \mathbf{x}) = (c_0 t, x, y, z)$$

of an event (relative to some frame $\mathcal{F}1$) make up a four-vector.

Notice that the spatial part $\mathbf{x} = (x, y, z)$ is the ordinary spatial position vector relative to $\mathcal{F}1$. We also make

Definition NN

The *four-momentum* of a particle with speed v, total energy E, and momentum $\mathbf{p} = (p_x, p_y, p_z)$ is

$$\vec{p} = \left(\frac{E}{c_0}, \mathbf{p}\right) = \left(\frac{E}{c_0}, p_x, p_y, p_z\right).$$

Theorem NN then reveals that

Corollary NN

The four-momentum of a particle is a four-vector.

Thus, from now on, we will write the four-coordinates as $\vec{X}$ and the four-momentum as $\vec{p}$. Notice that the spatial part of the four-momentum is the classical momentum.

Definition NN

A scalar quantity that is invariant under a Lorentz transformation (in other words, has the same value in any inertial frame of reference), is called a *Lorentz scalar*.

Examples of Lorentz scalars include the rest mass m and charge q of a particle and the speed of light c_0. However, p, p_x, p_y, E, x, y, z, u, u_x, u_y, … are clearly dependent upon the frame of reference, and are therefore not examples of Lorentz scalars. It is clear from the definition that any function of any number of Lorentz scalars is again a Lorentz scalar. For instance, the quantity $m^{c_0} \sinh q$ is a Lorentz scalar, as is the rest energy $E = mc_0^2$.

For future needs, we make

Proposition NN

$$\Lambda\eta = \eta\Lambda^{-1}.$$

Proof

$$\Lambda\eta = \begin{pmatrix} \gamma(v) & -\gamma(v)v/c_0 & 0 & 0 \\ -\gamma(v)v/c_0 & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \\ = \begin{pmatrix} \gamma(v) & \gamma(v)v/c_0 & 0 & 0 \\ -\gamma(v)v/c_0 & -\gamma(v) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

while (notice that Λ^{-1} is trivially found from Λ by making the substitution $v \to -v$)

$$\eta\Lambda^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma(v) & \gamma(v)v/c_0 & 0 & 0 \\ \gamma(v)v/c_0 & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\ = \begin{pmatrix} \gamma(v) & \gamma(v)v/c_0 & 0 & 0 \\ -\gamma(v)v/c_0 & -\gamma(v) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

■

Corollary NN

$$\Lambda\eta\Lambda = \eta.$$

3.6.3 Properties of Four-Vectors

The whole point of four-vectors is contained in the following result, which is almost immediate.

Theorem NN

Equality between four-vectors does not depend on the inertial frame of reference. That is, if $\vec{X}$ and $\vec{Y}$ are four-vectors relative to $\mathcal{F}1$, and $\overrightarrow{X'}$ and $\overrightarrow{Y'}$ are the corresponding four-vectors relative to $\mathcal{F}2$, then

$$\vec{X} = \vec{Y} \Rightarrow \overrightarrow{X'} = \overrightarrow{Y'}.$$

Let $\mathcal{F}1$ and $\mathcal{F}2$ be two different frames in standard configuration. Then the hypothesis

$$\vec{X} = \vec{Y} \Rightarrow \Lambda\vec{X} = \Lambda\vec{Y}$$

where Λ is the Lorentz transformation connecting $\mathcal{F}1$ to $\mathcal{F}2$. But since $\vec{X}$ is a four-vector,

$$\overrightarrow{X'} = \Lambda\vec{X}$$

and similarly for $\vec{Y}$. Thus

$$\vec{X} = \vec{Y} \Rightarrow \overrightarrow{X'} = \overrightarrow{Y'}.$$

■

Notice in particular how it is now obvious that the four-momentum conservation law

$$\overrightarrow{p_1} = \overrightarrow{p_2}$$

is compatible with the axioms of special relativity. Indeed, being an equality of four-vectors, if it is valid in some frame, then it is valid in any other frame. Since the components of the four-momentum are the energy (divided by a constant) and the relativistic momentum **p**, it is clear that both the law of energy conservation and the law of relativistic momentum conservation are compatible.

Notice also that is not a coincidence that we found out that $\vec{p}$ is a four-vector (in Theorem NN, essentially) while trying to display the compatibility of the aforementioned laws of conservation. Finally, notice that the 'entanglement' between space and time as illustrated by the transformation of the four-coordinates is *identical* to the entanglement between energy and momentum as illustrated by the transformation of the four-momentum.

At this point, the reader might object that the introduction of four-vectors doesn't really yield anything. Indeed, so far it has only given us *a more fancy way* of saying that a law of conservation is compatible with the Lorentz transformation. However, there is more than that to it. We will see that we can combine four-vectors into new four-vectors, and this will help us obtain many more results. In addition, this new language will help to bridge the gap to more advanced physical theories, such as the *general* theory of relativity

The following simple result will take us to the former benefit of four-vectors:

Proposition NN

Let $\vec{a}$ and $\vec{b}$ be four-vectors, and let k be a Lorentz scalar. Then

(1) $\vec{c} = \vec{a} + \vec{b}$ is a four-vector,

(2) $\vec{d} = k\vec{a}$ is a four-vector, and

(3) $\langle \vec{a}, \vec{b} \rangle$ is a Lorentz scalar.

If χ is a Lorentz scalar that may depend on time, and $\vec{a}$ is a four-vector that may depend on time too, then

(4) $\frac{d\vec{a}}{d\chi}$ is a four-vector.

Proof

Assume that $\vec{a}$ and $\vec{b}$ are four-vectors relative $\mathcal{F}1$, and let $\overrightarrow{a'}$ and $\overrightarrow{b'}$ be the corresponding four-vectors in $\mathcal{F}2$.

(1) and (2) are simple:

$$\Lambda\vec{c} = \Lambda\left(\vec{a} + \vec{b}\right) = \Lambda\vec{a} + \Lambda\vec{b} = \overrightarrow{a'} + \overrightarrow{b'} = \overrightarrow{c'},$$
$$\Lambda\vec{d} = \Lambda(k\vec{a}) = k\Lambda\vec{a} = k\overrightarrow{a'} = \overrightarrow{d'}.$$

When it comes to (3), we have

$$\langle \overrightarrow{a'}, \overrightarrow{b'} \rangle = \langle \Lambda\vec{a}, \Lambda\vec{b} \rangle = \left(\Lambda\vec{b}\right)^T \eta(\Lambda\vec{a}) = \vec{b}^T\Lambda^T\eta\Lambda\vec{a} = \vec{b}^T\Lambda\eta\Lambda\vec{a} = \vec{b}^T\eta\vec{a} = \langle \vec{a}, \vec{b} \rangle$$

using Corollary NN. Finally,

$$\Lambda\left(\frac{d\vec{a}}{d\chi}\right) = \Lambda\left(\lim_{\Delta\chi\to 0}\frac{\vec{a}(\chi+\Delta\chi)-\vec{a}(\chi)}{\Delta\chi}\right) = \lim_{\Delta\chi\to 0}\frac{\Lambda\vec{a}(\chi+\Delta\chi)-\Lambda\vec{a}(\chi)}{\Delta\chi} = \lim_{\Delta\chi\to 0}\frac{\overrightarrow{a'}(\chi+\Delta\chi)-\overrightarrow{a'}(\chi)}{\Delta\chi} = \\ = \frac{d\overrightarrow{a'}}{d\chi}.$$

■

Corollary NN

Any linear combination of four-vectors is a four-vector. In addition, the norm of any four-vector is a Lorentz scalar.

3.6.4 The Four-Vectors of SR Dynamics

We have seen that the four-coordinates $\vec{x}$ of an event (such as the spacetime position of a particle) is a four-vector. The four-coordinates generalise the radius (position) vector of Euclidean space $\mathbb{R}^3$ to spacetime. The analogue of the displacement vector $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the spacetime interval four-vector:

Definition NN

Let $\mathcal{F}1$ be some inertial frame, and consider two events $\vec{x} = (c_0 t, x, y, z)$ and $\vec{\tilde{x}} = (c_0\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$. The *(spacetime) separation* between the two events is the four-vector

$$\overrightarrow{\Delta x} \coloneqq \vec{\tilde{x}} - \vec{x} = (c_0\Delta t, \Delta x, \Delta y, \Delta z).$$

The spacetime separation is clearly a four-vector, since it is a linear combination of two four-vectors. Now, let us return to the four-coordinates of a single particle. The worldline is the image of the parameterisation function $\tau \mapsto \vec{x}(\tau)$, where τ is the proper time of the particle. By Proposition NN, the derivative $\frac{d\vec{x}}{d\tau}$ is too a four-vector. This is the four-velocity of the particle.

Definition NN

Let $\vec{x}$ be the four-coordinates of a particle, and let τ be the proper time parameter of the particle's worldline. Then

$$\vec{u} \coloneqq \frac{d\vec{x}}{d\tau}$$

is the *four-velocity* of the particle.

Corollary NN

Let $\vec{x} = (c_0 t, \mathbf{x})$ be the four-coordinates relative to some Minkowski frame $\mathcal{F}1$. Then the four-velocity, relative to this frame, is

$$\vec{u} = \gamma(u)(c_0, \mathbf{u})$$

where $\mathbf{u} = (u_x, u_y, u_z)$ is the usual three-velocity.

Proof

$$\vec{u} \stackrel{\text{def}}{=} \frac{d\vec{x}}{d\tau} = \frac{d}{d\tau}(c_0 t, x, y, z) = \left(\frac{d}{d\tau}(c_0 t), \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = \left(c_0 \frac{dt}{d\tau}, \frac{dx}{dt}\cdot\frac{dt}{d\tau}, \frac{dy}{dt}\cdot\frac{dt}{d\tau}, \frac{dz}{dt}\cdot\frac{dt}{d\tau}\right) =$$
$$= \left(c_0\gamma(u), u_x\gamma(u), u_y\gamma(u), u_z\gamma(u)\right) = \gamma(u)\left(c_0, u_x, u_y, u_z\right) = \gamma(u)(c_0, \mathbf{u})$$

■

Notice that the spatial part of the four-velocity is *not* the classical three-velocity $\mathbf{u}$, but rather $\gamma(u)\mathbf{u}$. Nevertheless, as one might almost expect, we have

Proposition NN

Let $\vec{u}$ be the four-velocity and $\vec{p}$ the four-momentum of a particle. Then

$$\vec{p} = m\vec{u}$$

where m is the rest mass of the particle.

Proof

Let $\mathcal{F}1$ be Minkowski coordinates. Then,

$$\vec{p} \stackrel{\text{def}}{=} \left(\frac{E}{c_0}, \mathbf{p}\right).$$

On the other hand,

$$m\vec{u} = m\gamma(u)(c_0, \mathbf{u}).$$

But

$$E \stackrel{\text{def}}{=} \gamma(u)mc_0^2, \qquad \mathbf{p} \stackrel{\text{def}}{=} \gamma(u)m\mathbf{u}$$

and so the proposed equality in $\mathcal{F}1$ is immediate. But since both the right-hand side and the left-hand side are four-vectors, equality holds in *any* inertial frame, and so the four-vector equation holds. ■

We could have *defined* the four-momentum as $m\vec{u}$. Had we done so, we would have been given the fact that the 4-tuple of numbers $\left(\frac{E}{c_0}, \mathbf{p}\right)$ is a four-vector for free, by Proposition NN, because m is a Lorentz scalar. In other words, the compatibility of the (relativistic) energy and momentum conservation laws would have been trivial. By now, it is irresistible to make

Definition NN

The four-acceleration of a particle with four momentum $\vec{p}$ is

$$\vec{a} \coloneqq \frac{d\vec{u}}{d\tau}.$$

The four-force on such a particle is

$$\vec{F} = \frac{d\vec{p}}{d\tau}.$$

For a moment, restrict attention to the common case where the rest mass of the particle remains constant (at least for the duration of the investigation of it). Since

$$\vec{p} \stackrel{\text{def}}{=} m\vec{u}$$

where m is a constant, we have, trivially,

$$\vec{F} \stackrel{\text{def}}{=} \frac{d\vec{p}}{d\tau} = \frac{d}{d\tau}(m\vec{u}) = m\frac{d\vec{u}}{d\tau} = m\vec{a}$$

precisely as in Newtonian mechanics, where $\mathbf{p} \stackrel{\text{def}}{=} m\mathbf{u}$ with constant m, too.

Corollary NN

Let $\mathcal{F}1$ be a Minkowski frame. Then

$$\vec{a} = \left(\frac{\gamma(u)^4(\mathbf{u}\cdot\mathbf{a})}{c_0}, \gamma(u)^2\mathbf{a} + \frac{\gamma(u)^4(\mathbf{u}\cdot\mathbf{a})}{c_0^2}\mathbf{u}\right)$$

and, assuming the rest mass m is constant,

$$\vec{F} = \left(\frac{\gamma(u)}{c_0}(\mathbf{F}\cdot\mathbf{u}), \gamma(u)\mathbf{F}\right)$$

where $\mathbf{F}$ is the relativistic force.

Proof

$$\begin{aligned}
\vec{a} = \frac{d\vec{u}}{d\tau} &= \frac{d}{d\tau}\big(\gamma(u)(c_0,\mathbf{u})\big) = \frac{d}{d\tau}\gamma(u)\cdot(c_0,\mathbf{u}) + \gamma(u)\frac{d}{d\tau}(c_0,\mathbf{u}) = \\
&= \frac{d}{du}\gamma(u)\cdot\frac{du}{dt}\cdot\frac{dt}{d\tau}\cdot(c_0,\mathbf{u}) + \gamma(u)\cdot\frac{d}{dt}(c_0,\mathbf{u})\cdot\frac{dt}{d\tau} = \\
&= \frac{u}{c_0^2\left(1-\frac{u^2}{c_0^2}\right)^{3/2}}\cdot\frac{\mathbf{u}\cdot\mathbf{a}}{u}\cdot\gamma(u)\cdot(c_0,\mathbf{u}) + \gamma(u)\cdot(0,\mathbf{a})\cdot\gamma(u) = \\
&= \frac{\gamma(u)^4(\mathbf{u}\cdot\mathbf{a})}{c_0^2}\cdot(c_0,\mathbf{u}) + \gamma(u)^2(0,\mathbf{a}) = \\
&= \left(\frac{\gamma(u)^4(\mathbf{u}\cdot\mathbf{a})}{c_0}, \gamma(u)^2\mathbf{a} + \frac{\gamma(u)^4(\mathbf{u}\cdot\mathbf{a})}{c_0^2}\mathbf{u}\right)
\end{aligned}$$

since

$$\frac{du}{dt} = \frac{d}{dt}|\mathbf{u}| = \frac{d}{dt}\sqrt{u_x^2+u_y^2+u_z^2} = \frac{1}{2}\left(u_x^2+u_y^2+u_z^2\right)^{-1/2}\cdot\left(2u_xa_x+2u_ya_y+2u_za_z\right) = \frac{\mathbf{u}\cdot\mathbf{a}}{u}.$$

Thus

$$\begin{aligned}
\vec{F} \stackrel{\text{def}}{=} \frac{d\vec{p}}{d\tau} = \frac{d}{d\tau}(m\vec{u}) = m\frac{d\vec{u}}{d\tau} &= \left(\frac{m\gamma(u)^4(\mathbf{u}\cdot\mathbf{a})}{c_0}, m\gamma(u)^2\mathbf{a} + \frac{m\gamma(u)^4(\mathbf{u}\cdot\mathbf{a})}{c_0^2}\mathbf{u}\right) = \\
&= \left(\frac{\gamma(u)}{c_0}[m\gamma(u)^3ua_{\parallel}], \gamma(u)\left[m\gamma(u)\mathbf{a} + \frac{m\gamma(u)^3(\mathbf{u}\cdot\mathbf{a})}{c_0^2}\mathbf{u}\right]\right) = \left(\frac{\gamma(u)}{c_0}[\mathbf{F}\cdot\mathbf{u}], \gamma(u)\mathbf{F}\right)
\end{aligned}$$

according to Proposition NN.

■

3.6.5 The Lorentz Scalars

According to Proposition NN and its corollary, the scalar product between any two four-vectors, and thus, in particular, the norm of a four-vector, is a Lorentz scalar. In the last subsection, we found a set of four-vectors, and so it is interesting to find out about the Lorentz scalars we can obtain from them.

3.6.5.1 The Four-Coordinates

Let us start with the four-coordinates,

$$\vec{x} = (c_0 t, x, y, z).$$

Its norm-square is

$$\|\vec{x}\|^2 = c_0^2 t^2 - x^2 - y^2 - z^2$$

and has to be a Lorentz scalar. That is, if relative to some other inertial frame $\mathcal{F}2$ with the same origin, the four-coordinates are

$$\overrightarrow{x'} = (c_0 t', x', y', z')$$

then the norm-square in this frame is

$$\left\|\overrightarrow{x'}\right\|^2 = c_0^2 t'^2 - x'^2 - y'^2 - z'^2$$

and

$$\|\vec{x}\| = \left\|\overrightarrow{x'}\right\|.$$

3.6.5.2 The Spacetime Separation

Similarly, the spacetime separation

$$\Delta\vec{x} = (c_0 \Delta t, \Delta x, \Delta y, \Delta z)$$

has norm-square

$$\left\|\overrightarrow{\Delta x}\right\|^2 = c_0^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

which is a Lorentz scalar. This is given a name of its own:

Definition NN

Let $\overrightarrow{\Delta x} = \vec{\tilde{x}} - \vec{x}$ be a spacetime separation. Then

$$\Delta S \coloneqq \left\|\overrightarrow{\Delta x}\right\|^2 = c_0^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

is called the *spacetime interval* between the two events $\vec{x}$ and $\vec{\tilde{x}}$.

That is, all observers agree on the spacetime interval between two events, even if they do not agree, in general, on the coordinates of the two events, or on the coordinates of their separation.

3.6.5.3 The Four-Velocity

The four-velocity is

$$\vec{u} = \gamma(u)(c_0, \mathbf{u})$$

with norm-square

$$\|\vec{u}\|^2 = \gamma(u)^2 c_0^2 - \gamma(u)^2\left(u_x^2 + u_y^2 + u_z^2\right) = \gamma(u)^2 c_0^2 - \gamma(u)^2 u^2 = \frac{c_0^2}{1-\frac{u^2}{c_0^2}} - \frac{u^2}{1-\frac{u^2}{c_0^2}} = \frac{c_0^2 - u^2}{1-\frac{u^2}{c_0^2}} =$$

$$= c_0^2 \frac{1-\frac{u^2}{c_0^2}}{1-\frac{u^2}{c_0^2}} = c_0^2.$$

That is, the speed of light is a Lorentz scalar, but – of course – we already knew that! Notice in particular that the 'four-speed', that is, the norm of the four-velocity, isn't just a Lorentz scalar: it is also constant! That is, *any* particle travels through spacetime with constant four-speed (namely, the speed of light). Therefore, by Section 3.6.1.2 and Definition NN, $\vec{u}$ is a unit tangent vector to the worldline of the particle.

3.6.5.4 The Four-Momentum

By definition, the four-momentum

$$\vec{p} = \left(\frac{E}{c_0}, \mathbf{p}\right)$$

so that

$$\|\vec{p}\|^2 = \frac{E^2}{c_0^2} - p^2 = \frac{\gamma(u)^2 m^2 c_0^4}{c_0^2} - \gamma(u)^2 m^2 u^2 = \gamma(u)^2 m^2 c_0^2 - \gamma(u)^2 m^2 u^2 = (c_0^2 - u^2)\gamma(u)^2 m^2 =$$
$$= c_0^2 m^2.$$

Thus, the norm of the four-momentum is

$$\|\vec{p}\| = c_0 m$$

which is a Lorentz scalar (again, as we already knew).

3.6.6 The Light Cone

In this subsection, we will introduce the important concepts of the light cone and causality. Consider an object located at a point $\mathbf{x}_0 = (x_0, y_0, z_0)$ in space emitting a flash of light in all directions at a time t_0. Thus, at any later time $t = t_0 + \Delta t > t_0$, the flash of light will make up a sphere S_t centred at $\mathbf{x}_0$ with radius $c_0\Delta t$. At this time, every observer inside S will know that the object has emitted a flash of light, while no observer outside S_t could possibly know this. Indeed, they have not yet been hit by the flash of light, and since no traveller can travel through space with a speed greater than c_0, no one could possibly outrun the flash and warn an outside observer. This implies a very fundamental statement about causality in spacetime:

Observation NN

No information can travel through space at a speed greater than c_0. In particular, given any observer in space, any event taking place a time Δt ago will be completely hidden to the observer if the distance $D = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$ from the observer (x_0, y_0, z_0) to the spatial position (x, y, z) of the event is greater than $c_0\Delta t$.

In the above, $x = x_0 + \Delta x$ and similarly for y and z. The equation for S_t in space is

$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = c_0^2(\Delta t)^2$$

which is a sphere of radius $c_0\Delta t$ centred about (x_0, y_0, z_0). Thus, in spacetime, S has equation

$$c_0^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = 0$$

which is a hypercone centred at the event $(c_0 t, x_0, y_0, z_0)$. Since it has codimension 1, it is a hypersurface of spacetime. Notice that the spacetime interval between the four-coordinates $\vec{x} = (c_0 t_0, x_0, y_0, z_0)$ and any point $\vec{x} + \overrightarrow{\Delta x} = (c_0(t + \Delta t), x + \Delta x, y + \Delta y, z + \Delta z) = (c_0 t, x, y, z)$ on S is

$$\Delta S \overset{\text{def}}{=} \left\|\overrightarrow{\Delta x}\right\| = c_0^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = 0.$$

Clearly, this is immediate from the fact that light travels with the speed of light, and serves as the motivation for the Minkowski inner product. Indeed, any spacetime separation four-vector $\overrightarrow{\Delta x}$ belongs to exactly one of the following three classes:

- Iff $\Delta S > 0$, the separation is called 'timelike'.
- Iff $\Delta S = 0$, the separation is called 'lightlike', or 'null'.
- Iff $\Delta S < 0$, the separation is called 'spacelike'.

It follows immediately that we have

Observation NN

The separation $\overrightarrow{\Delta x}$ between two points on the worldline of a material particle is timelike.

The separation $\overrightarrow{\Delta x}$ between two points on the worldline of a photon is lightlike.

If the separation between two spacetime events is spacelike, then any one of the events cannot *affect* the other event. In particular, it is impossible for any one of the events to be the *cause* of the other.

The hypercone

$$S\colon c_0^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = 0,$$

where $\Delta t = t - t_0$, $\Delta x = x - x_0$, and so on, centred about the event $\vec{x} = (c_0 t_0, x_0, y_0, z_0)$ is called the *light cone*, or the *null cone*, at $\vec{x}$. Its interior contains of all timelike separations starting at $\vec{x}$, its surface consists of all lightlike separations starting at $\vec{x}$, and its exterior consists of all spacelike separations starting at $\vec{x}$. We will now divide each of these three classes into two subclasses.

Let $\overrightarrow{\Delta x}$ be a separation, and let $\vec{t} = (1, 0, 0, 0)$ be the direction of time, which is a four-vector (check that). Then:

- Iff $\langle\overrightarrow{\Delta x}, \vec{t}\rangle > 0$, then $\overrightarrow{\Delta x}$ is 'future-pointing'.
- Iff $\langle\overrightarrow{\Delta x}, \vec{t}\rangle < 0$, then $\overrightarrow{\Delta x}$ is 'past-pointing'.

Notice that the projection $\langle\overrightarrow{\Delta x}, \vec{t}\rangle$ is simply the first (time) component of $\overrightarrow{(\Delta t)}$, just as if $\langle\cdot,\cdot\rangle$ has been the standard inner product on $\mathbb{R}^4$.

Human beings generally find it difficult to visualise subsets of $\mathbb{R}^4$. Therefore, in order to 'visualise' the light cone, we will 'suppress' one of the three spatial dimensions; that is, we will pretend

that space is two-dimensional instead of three-dimensional. Then spacetime, $M = \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$ becomes three-dimensional, and the hypercone reduces to a 2-cone

$$c_0^2(\Delta t)^2 = (\Delta x)^2 - (\Delta y)^2.$$

Since, from a qualitative point of view, the Euclidean plane and three-space are very similar in nature, this is actually a very fruitful technique of visualisation. Choose therefore any point $\vec{x}$ in spacetime, such as the event that represent you right now. Below is the light cone at $\vec{x}$ with one spatial dimension suppressed. For simplicity, we have set $c_0 = 1$ (alternatively, you can consider the vertical axis as being scaled).

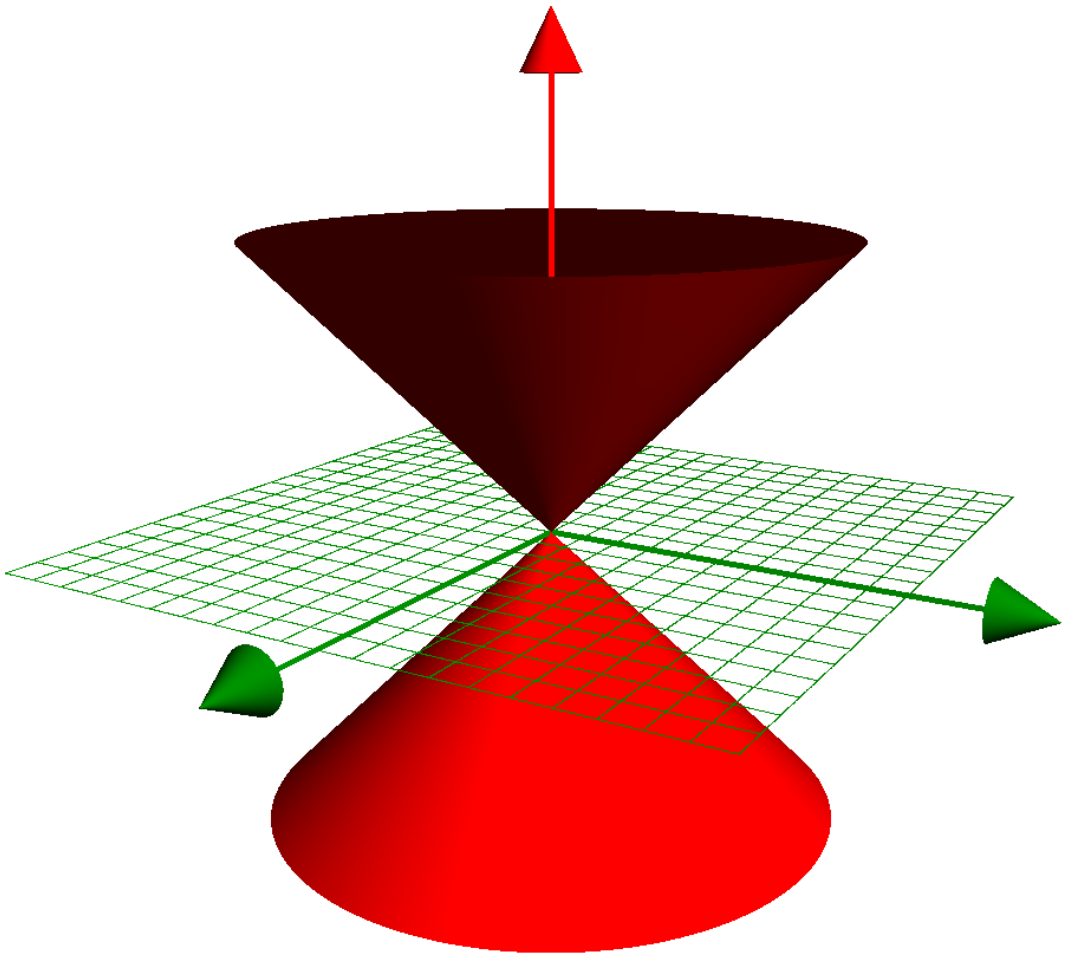


Figure 32. The light cone.

The vertical axis, corresponding to the red basis vector, is the time axis. That is, any 'slice' $t = \text{const}$ of spacetime, corresponding to a fixed time, is really the three-dimensional space at that time (that is, a *hyperplane* (and, of course, a *hypersurface*) in spacetime), but since we have suppressed one spatial dimension, it appears like a two-dimensional plane in the diagram above.

In particular, the slice $t = t_0$ is space at the current time of the event, indicated by the green plane in the diagram. The two green basis vectors are spatial basis vectors.

The upper 'half' of the cone is the *future light cone*, consisting of all events that a photon emitted at the vertex of the cone (that is, at the event $\overrightarrow{x_0}$). The intersection of the future light cone with any spatial slice $t = \text{const.}$ is a circle in the diagram above, but a sphere in reality. It is simply S_t that we met before. The interior of the future light cone is called the *future* of $\overrightarrow{x_0}$. The future consists of all points in spacetime that an observer located at $\overrightarrow{x_0}$ has a chance to visit. The union of the future light cone and the future of $\overrightarrow{x_0}$ is the set of spacetime events that could possible by affected by the event $\overrightarrow{x_0}$. Similarly, we define the *past light cone* as the lower 'half' of the light cone. This is the set of events $\mathcal{E}$ such that a photon emitted at $\mathcal{E}$ has a chance to reach $\overrightarrow{x_0}$ (if emitted in the right spatial direction). The interior of the past light cone is called the *past* of $\overrightarrow{x_0}$ and consists of all in spacetime in which an observer at $\overrightarrow{x_0}$ might have been at. The union of the past light cone and the past of $\overrightarrow{x_0}$ is the set of spacetime events that could possibly affect $\overrightarrow{x_0}$. The complement of $\{\overrightarrow{x_0}\} \cup$ light cone at $\overrightarrow{x_0} \cup$ future of $\overrightarrow{x_0} \cup$ past of $\overrightarrow{x_0}$ is called *elsewhere*. *Elsewhere* (which we will always italicise, due to risk of confusion with the adverb) consists of all spacetime events that can have no causal relationship with $\overrightarrow{x_0}$ whatsoever. This means that

- no event in *elsewhere* can affect $\overrightarrow{x_0}$ and
- $\overrightarrow{x_0}$ cannot affect any event in *elsewhere*.

If you think you have found a misprint in the paragraphs above, this is most likely due to a misunderstanding of the word 'event'. Since this is a rather common and easy-to-make mistake, we rephrase that

An event is a specification of a point in space AND a particular time.

Say that you are located at $\overrightarrow{x_0}$ and consider a different event $\vec{y}$ occurring at the same time (relative to your frame of reference) but a few meters (or even miles) away. Thus, if we draw the light cone at $\overrightarrow{x_0}$, then $\vec{y}$ might be the event indicated with a red dot in the diagram below:

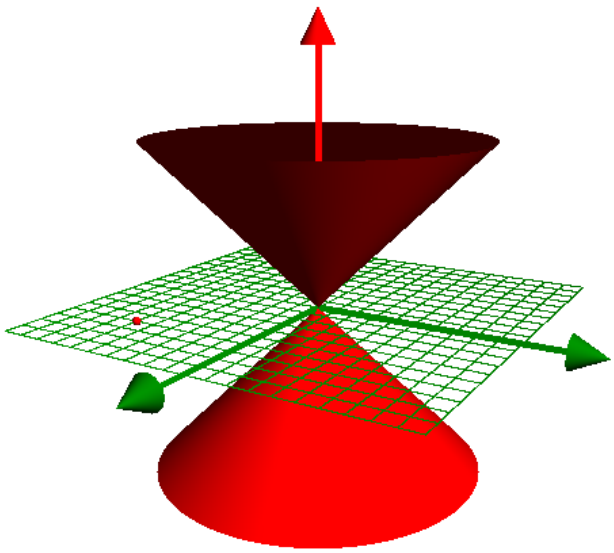


Figure 33. A light cone and an event in the hypersurface of 'the present'.

$\vec{y}$ clearly belongs to *elsewhere*, and, indeed, the spacetime separation between $\overrightarrow{x_0}$ and $\vec{y}$ is space-like, since $\Delta t = 0$ and $(\Delta x)^2 + (\Delta x)^2 + (\Delta y)^2 > 0$. Thus, the event $\vec{y}$ can have no influence on $\overrightarrow{x_0}$ whatsoever. But this does *not* mean that the happening at $\vec{y}$ can never affect you personally, it simply means that it cannot affect you at $\overrightarrow{x_0}$, that is, at time $t = t_0$. Indeed, a gas (nuclear) explosion happening in your kitchen (a town some hundred miles away) will not affect you, in your study, until several microseconds (seconds) later. That event, that is, you at that later time, is a *different* spacetime event (even if it is at the same point in space). And this spacetime event (which by the way is found in the future of $\overrightarrow{x_0}$, indeed, just 'above' $\overrightarrow{x_0}$ in the diagram) *is a member of the future of* $\vec{y}$, the light cone of which is *different* from the light cone of $\overrightarrow{x_0}$.

3.6.6.1 Transformation Properties of the Light Cone

Spacetime M is a geometric object, independent of any coordinate system, and so is a point, or event, $\vec{x}$ in it (even though, of course, its coordinates depends upon the observer's coordinate system). We have introduced the light cone at $\vec{x}$, which is a hypersurface of M. The question arises, "does the light cone at $\vec{x}$ depend upon the observer?" In other words, given a geometric point in spacetime (an event), does all possible observers agree upon the light cone at the event? In still other words, is 'the light cone at $\vec{x}$' well-defined as a geometric entity?

What do we need to show? We need to show that if some (inertial) observer considers an event $\vec{x}$ as a part of the light cone at $\overrightarrow{x_0}$, then any other (inertial) observer should agree. (The converse should also be true.) We will now show that it is.

Proof[45]

To this end, consider some inertial frame $\mathcal{F}1$ of reference, with a Minkowski coordinate system. We are interested in the light cone at $\overrightarrow{x_0} \in M$. Introduce some other Minkowski frame $\mathcal{F}2$ in standard configuration relative to $\mathcal{F}1$. Let $\overrightarrow{x_0'} = \Lambda x_0$ be the same geometric point as x_0, but expressed in $\mathcal{F}2$ coordinates. We will consider an *arbitrary* event in spacetime, known as $\vec{x}$ and $\overrightarrow{x'}$ relative to $\mathcal{F}1$ and $\mathcal{F}2$, respectively. Let the separation four-vectors be $\overrightarrow{\Delta x}$ and $\overrightarrow{\Delta x'}$, so that

$$\overrightarrow{x_0} + \overrightarrow{\Delta x} = \vec{x} \quad \text{and}$$
$$\overrightarrow{x_0'} + \overrightarrow{\Delta x'} = \overrightarrow{x'}.$$

Then it is clear, by the definition of the light cone inside each frame, that

$$\text{The event } \vec{x} \text{ belongs to the light cone at } \overrightarrow{x_0} \text{ relative to } \mathcal{F}1 \Leftrightarrow \left\|\overrightarrow{\Delta x}\right\| = 0 \quad \text{and}$$

$$\text{The event } \overrightarrow{x'} \text{ belongs to the light cone at } \overrightarrow{x_0'} \text{ relative to } \mathcal{F}2 \Leftrightarrow \left\|\overrightarrow{\Delta x'}\right\| = 0.$$

But $\overrightarrow{\Delta x}$ is a four-vector, and so its norm is a Lorentz scalar. Thus

$$\left\|\overrightarrow{\Delta x}\right\| = 0 \Leftrightarrow \left\|\overrightarrow{\Delta x'}\right\| = 0$$

which trivially implies

$$\text{The event } \vec{x} \text{ belongs to the light cone at } \overrightarrow{x_0} \text{ relative to } \mathcal{F}1 \Leftrightarrow$$
$$\Leftrightarrow \text{ The event } \overrightarrow{x'} \text{ belongs to the light cone at } \overrightarrow{x_0'} \text{ relative to } \mathcal{F}2$$

and we are done. ■

[45] You could argue that this proof is overly long, but I want it to be both easy-to-read and rigorous.

4 Classical Differential Geometry

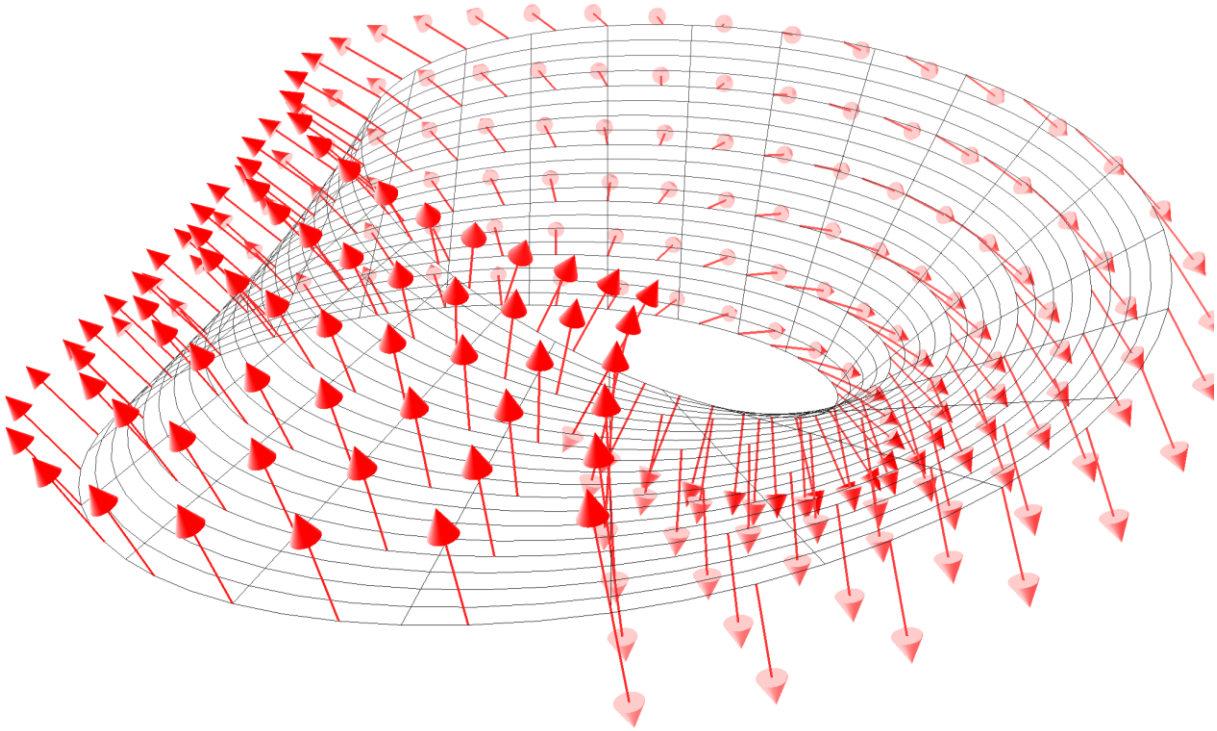


Figure 34. The Möbius band is a surface with only one side. In this chapter, we will explore curves and surfaces in ordinary space in order to prepare ourselves for a more general theory of 'manifolds' that is to come in the next chapter.

4.1 Introduction

This rather short chapter will be purely mathematical. It is included because the general theory of relativity, which we will turn to in the last chapters of the book, is formulated in the language of differential geometry. Indeed, we will find that spacetime curves in the presence of gravity, and that the curvature and metric properties of spacetime can be described in much the same language used to discuss the curvature of surfaces embedded in ordinary Euclidean space, which is far easier to understand.

In this chapter, we need the standard concepts

Definition NN

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$.

A continuous bijection $\phi: U \to V$ with a continuous inverse is called a *homeomorphism*. If such a function exists between U and V, then U and V are said to be *homeomorphic*. A differentiable bijection $\phi: U \to V$ with a differentiable inverse is called a *diffeomorphism*. If such a function exists, U and V are said to be *diffeomorphic*. If Φ is a diffeomorphism and both Φ and Φ^{-1} are smooth, then Φ is a *smooth diffeomorphism*.

and

Definition NN

Let

$$S^{n-1} \coloneqq \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$$

be the $(n-1)$-dimensional *unit sphere* in $\mathbb{R}^n$.

In particular,

$$S^1 \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is the unit circle and

$$S^2 \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

is the unit sphere.

We also make

Definition NN

Let

$$B^n \coloneqq \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\}$$

be the n-dimensional *(open) unit ball*.

Thus, B^3 is the usual, three-dimensional, open unit ball, B^2 is the open unit disk, and $B^1 =]-1, 1[$.

4.2 Curves

Although we are mainly interested in surfaces, we will start by considering curves.

4.2.1 What is a Curve?

It is difficult to give a precise definition of a 'curve' in a way that pleases everyone. In fact, it is hard to give a definition that entirely pleases the current author alone. The problem is that the word 'curve' is used in so many different, although related, ways, all of which seem highly natural. From a geometrical point of view, a curve is something you can draw with a pen on a paper. More precisely, it is the image $\mathbf{r}(I)$ of a continuous function $\mathbf{r}\colon I \to \mathbb{R}^2$ where I is an interval, probably, but not necessarily, bounded. Such a curve is thus a *set of points* $\mathbf{r}(I) \subset \mathbb{R}^2$. (For definitiveness, in this introductory subsection, we will only consider curves in the plane. The generalisation to curves in $\mathbb{R}^n$ causes almost no problems.)

From a physical point of view, we might want to consider the function $\mathbf{r}$ itself as the 'curve'. Indeed, $\mathbf{r}(t)$ might be the position of a particle at time $t \in I$. This is a different concept. In particular, there are generally many *different* functions $\mathbf{r}\colon I \to \mathbb{R}^2$, $\mathbf{q}\colon J \to \mathbb{R}^2$, ... with the same image $\mathbf{r}(I) = \mathbf{q}(J) = \cdots$. This can cause some trouble with the terminology if one is not careful. Say, for instance, that a particle moves two laps about the unit circle with unit speed starting at the origin of time, that is, its position is $\mathbf{r}(t) = (\cos t\,, \sin t)$ where $t \in [0, 4\pi]$. If you consider the curve to be the function, then the length of the curve is 4π. But the length of the 'point-set curve' $\mathbf{r}([0, 2\pi])$ is clearly only 2π, the same as the length of the 'curve' $\mathbf{r}(t) = (\cos t\,, \sin t)$ where $t \in [0, 2\pi]$. In addition, a curve, as a function, contains information about the *speed* of the particle at each point and the *orientation* of the curve. This information is not present in the image of the curve (the point set). Indeed, if $\mathbf{r}(t) = (\cos t\,, \sin t)$, $t \in I \coloneqq [0, 2\pi]$ and $\mathbf{q}(t) = (\cos 2t\,, \sin 2t)$, $t \in J \coloneqq [0, \pi]$ then $\mathbf{r}(I) = \mathbf{q}(J)$ although $|\mathbf{r}'(t)| \equiv 1$ while $|\mathbf{q}'(t)| \equiv 2$.

Irrespective if one considers a curve to be a function $\mathbf{r}$ on an interval I or the image $\mathbf{r}(I)$, one might want to require I to be either open or closed. If we demand that I is open, we can simplify many arguments and proofs since every point $t \in I$ looks like every other point; in other words, I does not contain any boundary points that might require special treatment. On the other hand, if we demand that I is closed, then we obtain many niceties if I is also bounded. Then $I = [a, b]$ and both $\mathbf{r}(a)$ and $\mathbf{r}(b) \in \mathbf{r}(I)$. Still, if we would require I to be either open or closed, a lot of functions/point sets that certainly looks like curves would not qualify for the term.

A curve, considered as a point set that you can draw by a pen on a paper, is said to be *closed* if it 'starts and ends at the same point'. This can be made precise, for instance, by saying that the curve is the image of a function $\mathbf{r}\colon [a, b] \to \mathbb{R}^2$ such that $\mathbf{r}(a) = \mathbf{r}(b)$. A curve is said to be *simple* if it does not intersect itself, that is, if $\mathbf{r}$ is *injective* except for the possibility $\mathbf{r}(a) = \mathbf{r}(b)$ if the curve is closed.

One might want to restrict the term 'curve' to mean only simple curves. This has a number of major advantages. For one thing, it would make the concepts of length of a function $\mathbf{r}\colon I \to \mathbb{R}^2$ and the length of a point set $\mathbf{r}(I)$ coincide (at the very least if you disregard pathological examples). Indeed, now $\mathbf{r}(t) = (\cos t\,, \sin t), t \in [0, 4\pi]$ is no longer a 'curve'. In addition, we will later introduce the concept of curvature, which measure the amount by which a 'curve curves' at some point $t \in I$. Intuitively, we wish the curvature to be a function $\kappa\colon \mathbf{r}(I) \to \mathbb{R}$. Indeed, if two curves $\mathbf{r}\colon I \to \mathbb{R}^2$ and $\mathbf{q}\colon J \to \mathbb{R}^2$ has the same image $\mathbf{r}(I) = \mathbf{q}(J)$, then the 'curvature' should be the same at each point on $\mathbf{r}(I)$ no matter if we use $\mathbf{r}$ or $\mathbf{q}$ to compute it. However, naturally, we *do* need to compute the curvature using some parameterisation function $\mathbf{r}$, and, unfortunately, it

is easy to see that, although we can define a curvature function $\kappa\colon I \to \mathbb{R}$ easily, in general, it is impossible to define a curvature function $\kappa\colon \mathbf{r}(I) \to \mathbb{R}$. The problem is that $\mathbf{r}$ might not be injective. There might be parameters $t, s \in I$, $t \neq s$, such that the curvatures $\kappa(t) \neq \kappa(s)$ while $\mathbf{r}(t) = \mathbf{r}(s)$, as illustrated below.

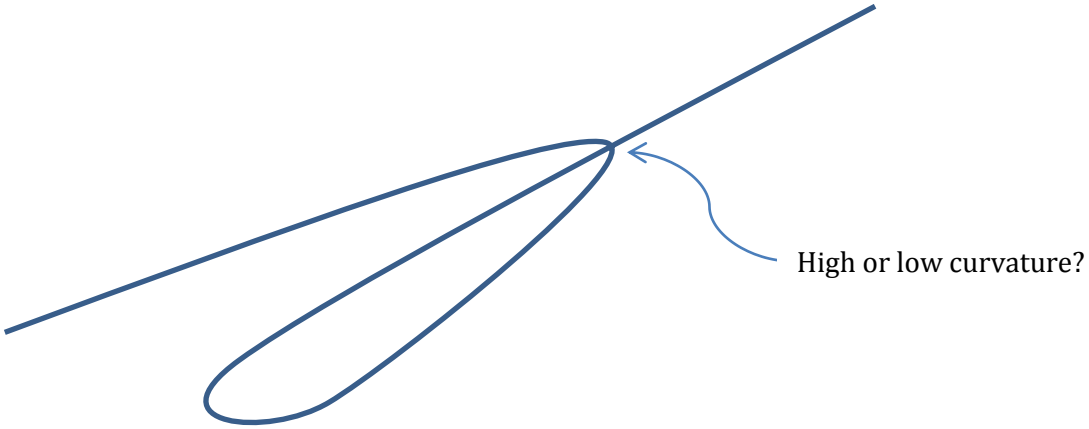


Figure 35. A non-simple curve with ambiguous curvature at the intersection.

This problem is obviously removed if we demand that the curve is simple, that is, if we demand that $\mathbf{r}\colon I \to \mathbb{R}^2$ is an injection. Then $\mathbf{r}$ is a bijection from I to its image $\mathbf{r}(I)$, and so $\kappa\colon \mathbf{r}(I) \to \mathbb{R}$ is certainly well-defined. Another nicety of simple curves is that every simple curve $\mathbf{r}(I)$ can be made into a totally ordered set by defining $\mathbf{p} < \mathbf{q} \Leftrightarrow r^{-1}(\mathbf{p}) < r^{-1}(\mathbf{q})$ where $r^{-1}\colon \mathbf{r}(I) \to I \subseteq \mathbb{R}$ is the inverse of $\mathbf{r}\colon I \to \mathbf{r}(I)$ which exists since $\mathbf{r}$ is injective. Finally, a simple curve has the nice property that is has no intersections. Although obvious, this is an important property in its own right when a curve is considered a 'manifold' (a concept we will define in later sections).

Unfortunately, we simply 'cannot' restrict our attention to simple curves, since we are interested in physics. Indeed, if $\mathbf{r}\colon I \to \mathbb{R}^2$ is the position function of a particle, then, of course, we must accept the possibility that $\mathbf{r}$ is not injective. In addition, even from a purely geometric point of view, it is natural to think of non-simple curves as being 'curves'.

With the above discussion in mind, it should be clear that we need to make some compromises when we define the 'curve'. We will use the word 'f-curve' ('f' as in 'function') to denote a function $I \to \mathbb{R}^n$, and the word 'curve' to denote the image of such a function. We will make no requirements on the interval I, and we will allow non-simple curves. In most cases, problems that arise due to non-injective f-curves can be removed by splitting the interval I into several smaller intervals, such that the function is injective on each interval. That's important to keep in mind.

Definition NN

A *curve parameterisation function*, or an *f-curve* for short, is a continuous function $\mathbf{r}\colon I \to \mathbb{R}^n$ where $I \subseteq \mathbb{R}$ is an interval of non-finite cardinality. The image $\gamma \coloneqq \mathbf{r}(I) \subset \mathbb{R}^n$ is called a *(parametric) curve*.

If I is an interval, let ∂I denote the set of boundary points of I [thus, $|I| \in \{0, 1, 2\}$]. Then let $\bar{I} \coloneqq I \cup \partial I$ be the *closure* of I [thus $\bar{I}$ is a closed set], and $\text{int}(I) \coloneqq I \setminus \partial I$ the *interior* of I [thus $\text{int}(I)$ is an open set].

If $n = 2$ (or $n = 3$) we speak of a plane curve (or a space curve).

4.2.2 Examples of Curves

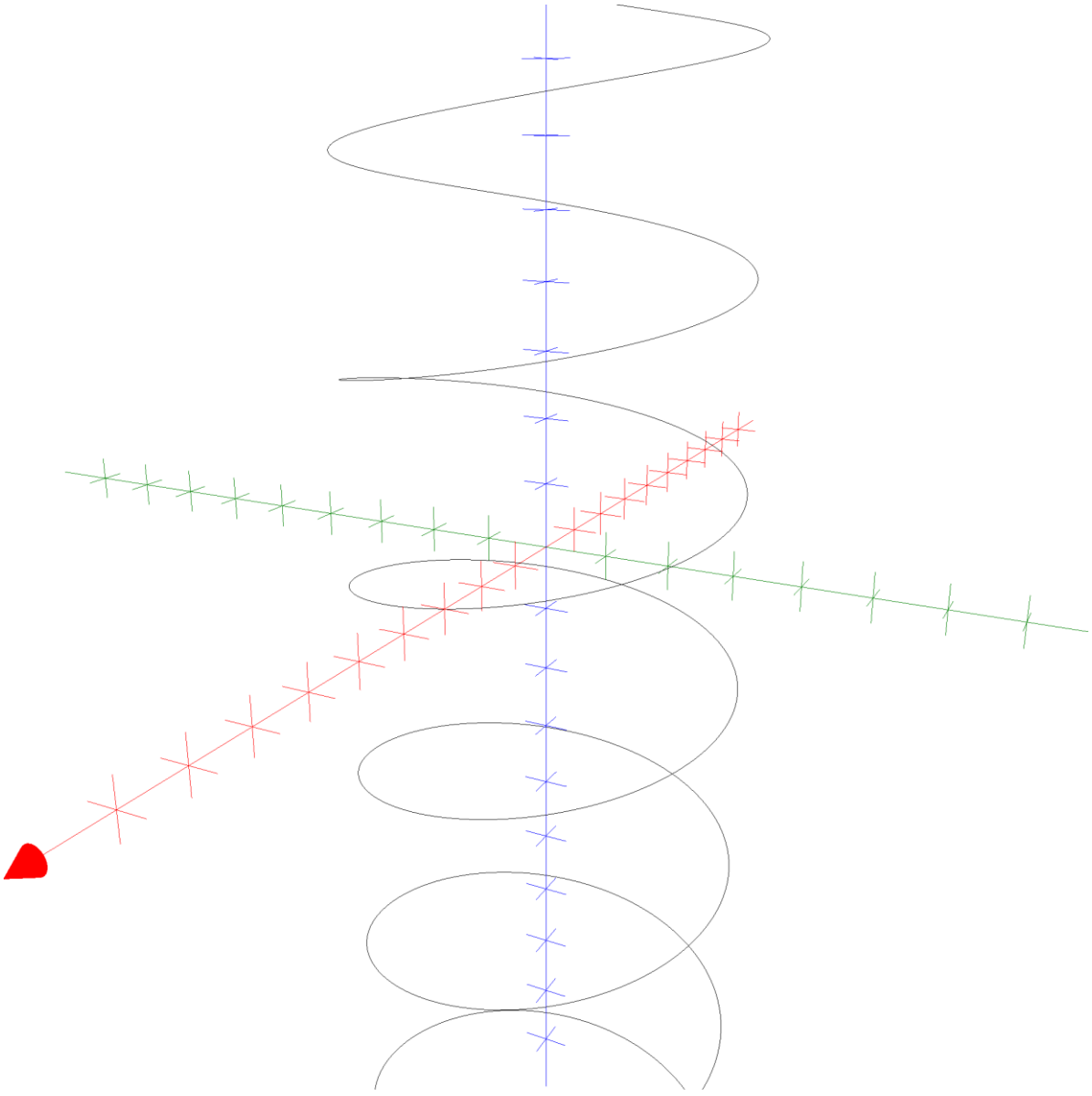
Example NN

The *circular (cylindrical) helix* of radius $a > 0$ and pitch $2\pi b > 0$ is the image $\gamma = \mathbf{r}(\mathbb{R})$ of

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} a\cos t \\ a\sin t \\ bt \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Obviously, the projection of the helix onto the xy-plane is a circle of radius a. Fix any point $\mathbf{p} \in \gamma$ on the helix and consider the smallest number $h > 0$ such that $\mathbf{p} + h\hat{\mathbf{z}} \in \gamma$. Clearly, the x and y coordinates of $\mathbf{p}$ and $\mathbf{p} + h\hat{\mathbf{z}}$ are both the same, and so t must be increased by 2π. Thus, the vertical distance between the points is $bt = 2\pi b$, that is, the pitch. Hence, the pitch is the vertical distance between successive points on the helix above the same point on the projection circle.

Below is a circular helix with parameters $a = 3$ and $b = 1/2$.

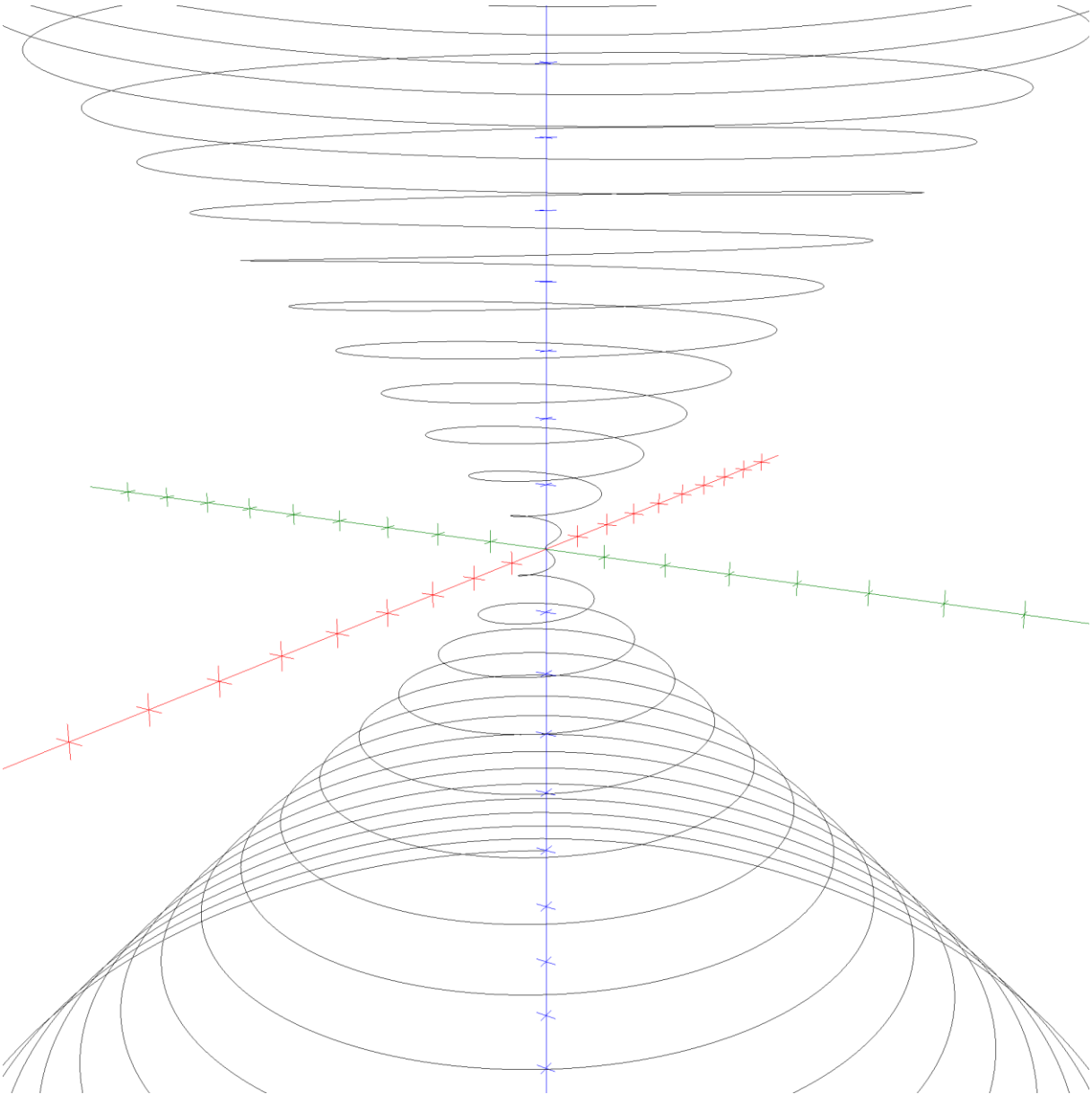


Example NN

The *circular conical helix* is the image $\gamma = \mathbf{r}(\mathbb{R})$ of

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} at\cos t \\ at\sin t \\ bt \end{pmatrix}, \qquad t \in \mathbb{R}$$

($a \neq 0, b \neq 0$). Below it is drawn for $a = b = 1/10$.



Example NN

A (straight) *line* passing through a point $\mathbf{x} \in \mathbb{R}^3$ with non-zero directional vector $\mathbf{v} \in \mathbb{R}^3$ is the image of

$$\mathbf{r}(t) = \mathbf{x} + t\mathbf{v}, \qquad t \in \mathbb{R}.$$

Many important curves are plane curves.

Example NN

Let $f\colon D \to \mathbb{R}$ be a function on $D \subseteq \mathbb{R}$. Then, by definition, its *graph* is

$$\{(x, y) \in \mathbb{R}^2 \colon (x \in D_f) \wedge (y = f(x))\}.$$

If f is continuous and D_f is an interval, then the graph is the parametric curve $\mathbf{r}(D_f)$ where

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} t \\ f(t) \end{pmatrix}, \qquad t \in D_f.$$

Example NN

A circle of radius a is the set of points $(x, y) \in \mathbb{R}^2$ satisfying $x^2 + y^2 = a^2$. This is a closed curve given by

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}, \qquad t \in [0, 2\pi].$$

An ellipse with semi axes lengths a and b is the set of points satisfying $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. This is a closed curve given by

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix}, \qquad t \in [0, 2\pi].$$

Example NN

The *Archimedean spiral* is the set of points satisfying the polar equation

$$r = a + b\varphi, \qquad a, b > 0.$$

The *logarithmic spiral* has the polar equation

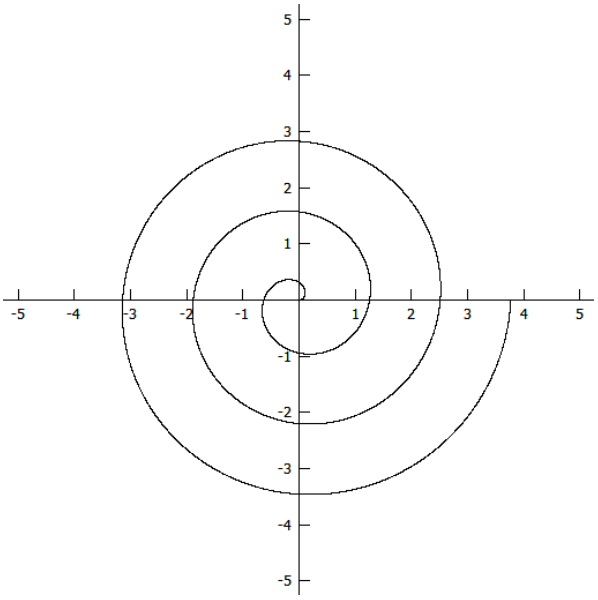
$$r = ae^{b\varphi}, \qquad a, b > 0.$$

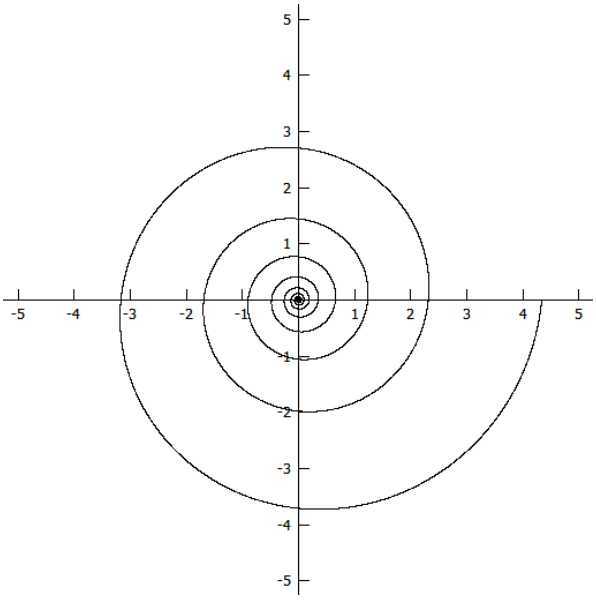
Both of these are parametric curves given by

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} r(t)\cos t \\ r(t)\sin t \end{pmatrix}$$

where $t \geq 0$ and $r(t) = a + bt$ in the first case, and $t \in \mathbb{R}$ and $r(t) = ae^{bt}$ in the latter case.

Below is the Archimedean spiral and the logarithmic spiral drawn for $(a, b) = \left(0, \frac{1}{5}\right), t \in [0, 6\pi]$ and $a = b = \frac{1}{10}, t \in]-\infty, 12\pi]$, respectively.





Example NN

The *butterfly curve* is the set of points described by the polar equation

$$r(\varphi) = e^{\sin\varphi} - 2\cos 4\varphi + \sin^5\frac{2\varphi - \pi}{24}, \qquad \varphi \in [0, 100].$$

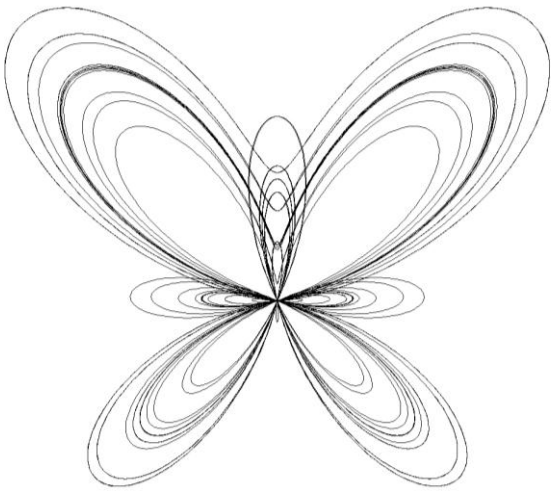
This is indeed a parameterised curve, given by

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} r(t)\cos t \\ r(t)\sin t \end{pmatrix}, \qquad t \in [0, 100]$$

(say) where

$$r(t) = e^{\sin t} - 2\cos 4t + \sin^5\frac{2t - \pi}{24}.$$

The butterfly curve is not a simple curve, as is obvious from the plot below.



We have seen examples of both plane curves and space curves. At this time, it is appropriate to make

Definition NN

Let $\gamma \subset \mathbb{R}^3$ be a space curve. If there exists a plane $\Pi \subset \mathbb{R}^3$ such that $\gamma \cap \Pi = \gamma$, then γ is *planar*.

Let $\gamma \subset \mathbb{R}^2$ be any plane graph. Then $\gamma \times \{c\} \subset \mathbb{R}^3$ is a planar space curve for every $c \in \mathbb{R}$. More generally, let A be any non-singular linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$, and B any spatial translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{d}$ ($\mathbf{d} \in \mathbb{R}^3$ constant). Then $B\big(A(\gamma \times \{0\})\big)$ is a planar space curve.

4.2.3 Some Technical Notes on Parameterisation

In order to distinguish 'nice' f-curves and curves from 'pathological' ones, we make

Definition NN

An f-curve $\mathbf{r}\colon I \to \mathbb{R}^n$ is *regular* iff it is smooth and $\mathbf{r}'(t) \neq \mathbf{0}$ for every $t \in I$. A curve is smooth (or regular) iff it is the image of some smooth (or regular) f-curve.

An f-curve $\mathbf{r}$ might be thought of giving the position of a particle at a given time. Hence, the derivative $\mathbf{r}'(t)$ might be thought of as the *velocity* at t. In particular, the magnitude $|\mathbf{r}'(t)|$ can be thought of as the *speed* at t. This motivates

Definition NN

An f-curve $\mathbf{r}\colon I \to \mathbb{R}^n$ is *unit-speed* if $|\mathbf{r}(t)| = 1$ for all $t \in I$.

It is obvious that a given curve γ in general may be the image of many different f-curves. The unit circle, for instance, is the image of infinitely many f-curves. Some traverse the circle once, and some traverse it several times. In addition, different f-curves may have different speeds and orientations. For instance, the f-curve

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, \qquad t \in I = [0, \pi]$$

has the same image as the f-curve

$$\mathbf{q}(t) = \underline{\mathbf{e}}\begin{pmatrix}\cos 2t\\ \sin 2t\end{pmatrix}, \qquad t \in J = \left[0, \frac{\pi}{2}\right],$$

namely, the upper-half unit circle. Notice that $\mathbf{r}$ is unit-speed, while $\mathbf{q}$ is not. Notice also that there exists an increasing smooth diffeomorphism $\Phi\colon I \to J$ such that $\mathbf{r}(t) = \mathbf{q}\big(\Phi(t)\big)$ for all $t \in I$. Indeed, $\Phi(t) = \frac{1}{2}t$. Thus, Φ can be used to translate the 'I-coordinate' of a point on $\mathbf{r}(I) = \mathbf{q}(J)$ to the corresponding 'J-coordinate'. In addition, Φ^{-1} exists and $\mathbf{q}(t) = \mathbf{r}\big(\Phi^{-1}(t)\big)$ for all $t \in J$: $\Phi^{-1}(t) = 2t$ and translates 'J-coordinates' to 'I-coordinates'. This motivates

Definition NN

Let $\mathbf{r}\colon I \to \mathbb{R}^n$ be an f-curve. An f-curve $\mathbf{q}\colon J \to \mathbb{R}^n$ is called a *reparameterisation* of $\mathbf{r}$ iff there exists an increasing smooth diffeomorphism $\Phi\colon I \to J$ such that $\mathbf{r}(t) = \mathbf{q}\big(\Phi(t)\big)$ for all $t \in I$. Φ is called the *reparameterisation map* from $\mathbf{r}$ to $\mathbf{q}$.

Lemma NN

Let $\mathbf{r}: I \to \mathbb{R}^n$ be an f-curve and let $\mathbf{q}: J \to \mathbb{R}^n$ be a reparameterisation of $\mathbf{r}$ with reparameterisation map Φ. Then

(1) $\Phi'(t) \neq 0$ for every $t \in I$.

(2) $\mathbf{r}$ is a reparameterisation of $\mathbf{q}$ with reparameterisation map Φ^{-1},

(3) if $\mathbf{r}$ is regular then $\mathbf{q}$ is regular,

(4) $\mathbf{r}(I) = \mathbf{q}(J)$,

(5) $|\mathbf{r}'(t)| = |\mathbf{q}'(\Phi(t)| \cdot \Phi'(t)$, and

Proof

(1) Since Φ is a diffeomorphism, $(\Phi^{-1})'\big(\Phi(t)\big) = 1/\Phi'(t)$ exists for every $t \in I$. Thus $\Phi'(t) \neq 0$ everywhere.

(2) Since Φ is a smooth diffeomorphism, its inverse Φ^{-1} exists and is a smooth diffeomorphism, too, and $\Phi^{-1}: J \to I$. In addition, pick any $t \in I$ and let $t' = \Phi(t) \in J$. Then the defining equation $\mathbf{r}(t) = \mathbf{q}\big(\Phi(t)\big)$ implies $\mathbf{r}\big(\Phi^{-1}(t')\big) = \mathbf{q}(t')$, and, since $\Phi(I) = J$, this holds for every $t' \in J$.

(3) Assume $\mathbf{r}$ is regular. Then the defining equation $\mathbf{q}(t) = \mathbf{r}\big(\Phi^{-1}(t)\big)$ gives $\mathbf{q}(t)$ as the composition of two smooth functions $\Phi^{-1}: J \to I$ and $\mathbf{r}: I \to \mathbb{R}^n$. Thus $\mathbf{q}: J \to \mathbb{R}^n$ is smooth. In addition, for every $t \in J$, $\mathbf{q}'(t) = \mathbf{r}'\big(\Phi^{-1}(t)\big) \cdot (\Phi^{-1})'(t)$ where $\mathbf{r}'\big(\Phi^{-1}(t)\big) \neq \mathbf{0}$ because $\mathbf{r}$ is regular, and $(\Phi^{-1})'(t) \neq 0$ according to (1) and (2). Thus $\mathbf{q}$ is smooth and $\mathbf{q}'(t) \neq 0$ and so, by definition, $\mathbf{q}$ is regular.

(4) The defining equation $\mathbf{r}(t) = \mathbf{q}\big(\Phi(t)\big)$ yields $\mathbf{r}(I) = \mathbf{q}(J)$ since $J = \Phi(I)$.

(5) This follows immediately from the defining equation and the chain rule. ■

It should be clear that $\mathbf{r}$ and $\mathbf{q}$, if they are reparameterisations of each other, have very similar properties. Not 'only' do they have the same image, as the following example shows.

Example NN

It is *not* true that every pair of f-curves with the same image are reparameterisations of each other. For instance, $\mathbf{r}(t) = (\cos t\,, \sin t), t \in I \coloneqq [0, 4\pi]$ and $\mathbf{r}(t) = (\cos t\,, \sin t), t \in J \coloneqq [0, 2\pi]$ clearly have the same image $\mathbf{r}(I) = \mathbf{q}(J) = S^1$, but there does not exists a bijection $\Phi: I \to J$ such that $\mathbf{r}(t) = \mathbf{q}\big(\Phi(t)\big)$ for all $t \in I$. Indeed, if there is a function $\Phi: I \to J$ such that$\mathbf{r}(t) = \mathbf{q}\big(\Phi(t)\big)$ for all $t \in I$, then, since $\mathbf{r}(\pi) = \mathbf{r}(3\pi) = (-1, 0)$, we must have $\Phi(\pi) = \Phi(3\pi) = \pi$ because only π gets mapped to $(-1, 0)$ by $\mathbf{q}$. But then Φ is not injective.

One might say that a reparameterisation $\mathbf{q}$ of an f-curve $\mathbf{r}$ preserves all physical properties of the trajectory $\mathbf{r}$ except for its speed at 'corresponding' points, which is altered according to Lemma NN(5). In particular, if $\mathbf{r}$ traverses the unit circle k times, then so does $\mathbf{q}$.

Lemma NN

Let $\mathbf{r}\colon I \to \mathbb{R}^n$ and $\mathbf{q}\colon J \to \mathbb{R}^n$ be two injective regular curves with the same image $\gamma = \mathbf{r}(I) = \mathbf{q}(J)$, which is a simple regular curve. Assuming that $\mathbf{r}$ and $\mathbf{q}$ are smooth diffeomorphisms to γ, there exists a reparameterisation map $\Phi\colon I \to J$ from $\mathbf{r}$ to $\mathbf{q}$.

Proof

Since $\mathbf{r}$ is a smooth diffeomorphism, its inverse $r^{-1}\colon \gamma \to I$ exists and is smooth, and the same applies for the inverse $q^{-1}\colon \gamma \to J$ of $\mathbf{q}$. Define $\Phi(t) \coloneqq q^{-1}\big(\mathbf{r}(t)\big)$ for all $t \in I$; then $\Phi(I) = J$ and since both $\mathbf{r}$ and q^{-1} are smooth diffeomorphisms, so is Φ. In addition, application of $\mathbf{q}$ yields $\mathbf{q}\big(\Phi(t)\big) = \mathbf{r}(t)$ which is the defining equation of a reparameterisation map. ■

4.2.4 The Length of an f-Curve

We make the obvious

Definition NN

The *length* of a regular f-curve $\mathbf{r}\colon I \to \mathbb{R}^n$ is

$$L \coloneqq \int_I |\mathbf{r}'(t)|dt.$$

The *signed length* of the part of the curve from $t = a$ to $t = b$ (where $a, b \in \bar{I}$ or possibly $\pm\infty$) is

$$L \coloneqq \int_a^b |\mathbf{r}'(t)|dt.$$

Notice that, by definition, the length is a property of an f-curve, not of a curve. Thus, the f-curve $\mathbf{r}(t) = (\cos t\,, \sin t), t \in I \coloneqq [0, 4\pi]$ has length 4π, although the curve $\mathbf{r}(I) = S^1$ has length 2π. Even though it is essentially apparent from the geometrical ideas that lead to Definition NN (which the reader is supposed to know very well), we might want to check some fundamental niceties:

Corollary NN

Let $\mathbf{r}\colon I \to \mathbb{R}^n$ be a regular f-curve and $\mathbf{q}\colon J \to \mathbb{R}^n$ a reparameterisation of $\mathbf{r}$. Then

$$\int_I |\mathbf{r}'(t)|dt = \int_J |\mathbf{q}'(t)|dt.$$

Proof

By Lemma NN,

$$\int_I |\mathbf{r}'(t)|dt = \int_I \big|\mathbf{q}'\big(\Phi(t)\big)\big| \cdot \Phi'(t)dt = \begin{bmatrix} t' = \Phi(t) \\ dt' = \Phi'(t)dt \end{bmatrix} = \int_J |\mathbf{q}'(t')|dt'.$$

■

Slightly more generally, we have

Corollary NN

Let $\mathbf{r}: I \to \mathbb{R}^n$ and $\mathbf{q}: J \to \mathbb{R}^n$ be two injective regular f-curves with the same image $\gamma = \mathbf{r}(I) = \mathbf{q}(J)$. Assuming that $\mathbf{r}$ and $\mathbf{q}$ are smooth diffeomorphisms to γ, the length of $\mathbf{r}$ is the same as the length of $\mathbf{q}$, that is, the length of a simple regular curve γ does not depend on its (smooth) parameterisation.

Proof

According to Proposition NN, there exists a reparameterisation map $\Phi: I \to J$ from $\mathbf{r}$ to $\mathbf{q}$. Thus, using Corollary NN, the lengths of $\mathbf{r}$ and $\mathbf{q}$ coinside. ■

If a curve is not simple, but intersects in a finite number of points (such as the butterfly curve shown above), the length is still independent of parameterisation, which you can show by partitioning I into smaller intervals such that the f-curve is injective in each interval. We will not go into the details.

Example NN

Consider the (circular elliptical) helix γ from Example NN:

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} a\cos t \\ a\sin t \\ bt \end{pmatrix}, \qquad t \in \mathbb{R}.$$

The restriction of $\mathbf{r}$ to $]t_0, t_0 + 2\pi[$ is an f-curve, the image of which is a single 'turn' of the helix γ. Since

$$\mathbf{r}'(t) = \underline{\mathbf{e}}\begin{pmatrix} -a\sin t \\ a\cos t \\ b \end{pmatrix}, \qquad |\mathbf{r}'(t)| = \sqrt{a^2 + b^2}$$

the length of such a 'turn' is

$$\int_{t_0}^{t_0+2\pi} \sqrt{a^2 + b^2}\,dt = 2\pi\sqrt{a^2 + b^2}.$$

Example NN

We want to find the lengths of the spirals drawn in Example NN. The Archimedean spiral is parameterised by

$$\mathbf{r}(t) = \frac{1}{5}\underline{\mathbf{e}}\begin{pmatrix} t\cos t \\ t\sin t \end{pmatrix}, \qquad t \in [0, 6\pi]$$

Thus

$$\mathbf{r}'(t) = \frac{1}{5}\underline{\mathbf{e}}\begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \end{pmatrix}, \qquad |\mathbf{r}'(t)| = \frac{1}{5}\sqrt{1 + t^2}$$

so that the length

$$L = \frac{1}{5}\int_0^{6\pi} \sqrt{1+t^2}dt \approx 35.9.$$

The logarithmic spiral is parameterised by

$$\mathbf{r}(t) = \frac{1}{10}\underline{\mathbf{e}}\begin{pmatrix} e^{t/10}\cos t \\ e^{t/10}\sin t \end{pmatrix}, \qquad t \in]-\infty, 12\pi]$$

and so

$$\mathbf{r}'(t) = \frac{1}{10}e^{t/10}\underline{\mathbf{e}}\begin{pmatrix} \frac{1}{10}\cos t - \sin t \\ \frac{1}{10}\sin t + \cos t \end{pmatrix}, \qquad |\mathbf{r}'(t)| = \frac{1}{10}e^{t/10}\sqrt{\frac{101}{100}}$$

and

$$L = \frac{\sqrt{101}}{100}\int_{-\infty}^{12\pi} e^{t/10}dt \approx 43.6.$$

Consider a regular f-curve $\mathbf{r}\colon I \to \mathbb{R}^n$. Fix any $t_0 \in I$ and consider the signed distance $s(t)$ from $t_0 \in \bar{I}$ to $t \in \bar{I}$. Intuitively, s assigned a unique number to each point on $\gamma = \mathbf{r}(I)$, namely, the signed distance from $\mathbf{r}(t_0)$ along the curve, and so can be used as a parameter, called an *arc-length parameter*. Quantitatively,

$$s(t) = \int_{t_0}^{t} |\mathbf{r}'(t')|dt'$$

so that

$$\frac{ds}{dt} = |\mathbf{r}'(t)| > 0, \qquad \forall t \in I$$

because $\mathbf{r}$ is regular. Hence s is a strictly increasing function and a diffeomorphism $I \to J = s(I)$ where J is also an interval. Thus, $s \mapsto t$ is a reparameterisation map, and s is a valid parameter. The arc-length parameter is not unique, however. If s and s' are two arc-length parameters, then $s = s' + k$ for some constant k. Indeed, if s measures arc length from t_0 and s' measures arc length from t_0', then

$$s(t) - s'(t) = \int_{t_0}^{t} |\mathbf{r}'(t')|dt' - \int_{t_0'}^{t} |\mathbf{r}'(t')|dt' = \int_{t_0}^{t_0'} |\mathbf{r}'(t')|dt' = \text{const.}$$

Definition NN

Let γ be a curve given by a regular $\mathbf{r}\colon I \to \mathbb{R}^n$. Choose any $t_0 \in I$ and introduce the arc-length parameter

$$s(t) = \int_{t_0}^{t} |\mathbf{r}'(t')|dt'$$

that measures the signed arc length from t_0. Then any parameter $s' = s + k$, where $k \in \mathbb{R}$ is arbitrary, is also called an *arc-length* parameter.

We state now the important relation between the arc-length parameter and the unit-speed parameterisation:

Proposition NN

Let $\mathbf{r}: I \to \mathbb{R}^n$ be a regular f-curve. Then $\mathbf{r}$ is unit-speed if and only if the parameter is an arc-length parameter.

Proof

Suppose that the parameterisation is unit-speed. Then the arc-length from $t_0 \in I$ to $t \in I$ is

$$s(t) = \int_{t_0}^{t} |\mathbf{r}'(t')| dt' = \int_{t_0}^{t} dt' = t - t_0,$$

that is, t is an arc-length parameter. Conversely, if t is an arc-length parameter, then the signed distance from $t_0 \in I$ to $t \in I$ is

$$\int_{t_0}^{t} |\mathbf{r}'(t')| dt' = t + k$$

for some constant k. Differentiation with respect to t yields

$$|\mathbf{r}'(t)| = 1$$

and so $\mathbf{r}$ is unit-speed. ■

Since every regular function has an arc-length parameter, we have

Corollary NN

Every regular curve has a unit-speed reparameterisation function.

In what follows, we will need

Lemma NN

Let γ be the image of a regular, unit-speed parameterisation function $\mathbf{r}: I \to \mathbb{R}^n$. Then

$$\mathbf{r}'(t) \perp \mathbf{r}''(t), \qquad \forall t \in I.$$

Proof

By hypothesis,

$$\big(\mathbf{r}'(t)\big)^2 = 1.$$

Differentiation w.r.t. t yields

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0.$$

■

4.2.5 Curvature

We wish to introduce a quantity that measures how much an f-curve $\mathbf{r}\colon I \to \mathbb{R}^n$ 'curves', that is, how much it deviates from a straight line. Intuitively, we wish this quantity to be a non-negative number defined at every $t \in I$, and a line should have zero curvature everywhere, while a circle of radius a should have the same curvature at every point, and this should be a decreasing function of a. Essentially, we would like the curvature to be a function on the curve, that is, on the set $\mathbf{r}(I)$. However, if the curve is not simple, that is, if $\mathbf{r}$ is not injective, and this might be a slight problem as discussed in the introduction of this section. Nevertheless, we require that the curvature be independent on parameterisation *as far as possible*. For instance, in a simple curve, the curvature should be a function defined on the curve, irrespective of parameterisation.

A natural approach would be to use the magnitude of the *second derivative* of the parameterisation function. However, such a concept would not be suitable at all: it would not only depend on the curve γ, but also on the parameterisation function $\mathbf{r}$, even if we only take simple curves and injective functions into account. For example, the circle parameterisations $(\cos t\,, \sin t), t \in [0, 2\pi]$ and $(\cos 2t\,, \sin 2t), t \in [0, \pi]$ would yield two different curvatures. If we demand that the parameterisation be unit-speed, however, then it will work.

Definition NN

Let $\mathbf{r}\colon I \to \mathbb{R}^n$ be a unit-speed regular f-curve. Then the *curvature* of $\mathbf{r}$ at $t \in I$ is

$$\kappa := |\mathbf{r}''(t)|.$$

Let $\mathbf{r}\colon I \to \mathbb{R}^n$ be any regular f-curve with a unit-speed reparameterisation $\mathbf{q}\colon J \to \mathbb{R}^n$ and reparameterisation map $\Phi\colon I \to J$. Then the *curvature* of $\mathbf{r}$ at $t \in I$ is defined as the curvature of $\mathbf{q}$ at $\Phi(t) \in J$.

We need to check that the latter concept is well-defined, that is, that the curvature of $\mathbf{r}$ does not depend upon *which* unit-speed reparameterisation function $\mathbf{q}$ you choose to use for the curvature computation.

Proposition NN

Let $\mathbf{r}\colon I \to \mathbb{R}^n$ be a regular f-curve with two unit-speed reparameterisation functions $\mathbf{q}\colon J \to \mathbb{R}^n$ and $\mathbf{p}\colon K \to \mathbb{R}^n$ with corresponding reparameterisation maps $\Phi\colon I \to J$ and $\Psi\colon I \to K$. Then

$$\big|\mathbf{q}''\big(\Phi(t)\big)\big| = \big|\mathbf{p}''\big(\Psi(t)\big)\big|, \qquad \forall t \in I.$$

Proof

We have

$$\mathbf{r}(t) = \mathbf{q}\big(\Phi(t)\big) = \mathbf{p}\big(\Psi(t)\big),$$

but since both $\Phi(t)$ and $\Psi(t)$ are arc-length parameters,

$$\Psi(t) = \Phi(t) + k$$

for some $k \in \mathbb{R}$. Thus

$$\mathbf{q}\big(\Phi(t)\big) = \mathbf{p}(\Phi(t) + k).$$

Differentiation with respect to t yields

$$\mathbf{q}'\big(\Phi(t)\big)\Phi'(t) = \mathbf{p}'(\Phi(t)+k)\Phi'(t)$$

and, since $\Phi'(t) \neq 0$,

$$\mathbf{q}'\big(\Phi(t)\big) = \mathbf{p}'(\Phi(t)+k).$$

Differentiating again,

$$\mathbf{q}''\big(\Phi(t)\big)\cdot\Phi'(t) = \mathbf{p}''(\Phi(t)+k)\cdot\Phi'(t)$$

and

$$\mathbf{q}''\big(\Phi(t)\big) = \mathbf{p}''(\Phi(t)+k) = \mathbf{p}''\big(\Psi(t)\big).$$

Thus

$$\big|\mathbf{q}''\big(\Phi(t)\big)\big| = \big|\mathbf{p}''\big(\Psi(t)\big)\big|.$$

■

Example NN

Let γ be a straight line given by

$$\mathbf{r}(t) = \mathbf{x} + t\mathbf{v}, \qquad t \in \mathbb{R}$$

where $\mathbf{x}, \mathbf{v} \in \mathbb{R}^n$, $|\mathbf{v}| = 1$. Then $|\mathbf{r}'(t)| = |\mathbf{v}| = 1$ so that $\mathbf{r}$ is unit-speed. Since $\mathbf{r}'' = \mathbf{0}$, the curvature

$$\kappa \stackrel{\text{def}}{=} |\mathbf{r}''(t)| = 0, \qquad \forall t \in I.$$

Example NN

Let γ be a part of a circle of radius a, given by

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} a\cos(t/a) \\ a\sin(t/a)\end{pmatrix}, \qquad t \in I.$$

Since

$$\mathbf{r}'(t) = \underline{\mathbf{e}}\begin{pmatrix} -\sin(t/a) \\ \cos(t/a)\end{pmatrix}, \qquad |\mathbf{r}'(t)| = 1,$$

$\mathbf{r}$ is unit-speed. Furthermore,

$$\mathbf{r}''(t) = \frac{1}{a}\underline{\mathbf{e}}\begin{pmatrix} -\cos(t/a) \\ -\sin(t/a)\end{pmatrix}$$

so that the curvature

$$\kappa \stackrel{\text{def}}{=} |\mathbf{r}''(t)| = \frac{1}{a}.$$

The above examples show that our definition satisfies our demands. Now we can obtain some more interesting results.

Example NN

The (circular cylindrical) helix from Example NN is given by

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} a\cos t \\ a\sin t \\ bt \end{pmatrix}, \qquad t \in \mathbb{R}.$$

This is not unit-speed, since

$$|\mathbf{r}'(t)| = \sqrt{a^2+b^2} \neq 1 \quad \text{in general.}$$

We thus need to 'slow it down' a bit. It is clear that

$$\mathbf{r}(t) = \underline{\mathbf{e}}\begin{pmatrix} a\cos(t/v) \\ a\sin(t/v) \\ b(t/v) \end{pmatrix}, \qquad t \in \mathbb{R},$$

where $v \coloneqq \sqrt{a^2+b^2}$, is a unit-speed parameterisation for γ. Indeed,

$$\mathbf{r}'(t) = \frac{1}{v}\underline{\mathbf{e}}\begin{pmatrix} -a\sin(t/v) \\ a\cos(t/v) \\ b \end{pmatrix}, \qquad |\mathbf{r}'(t)| = \frac{1}{v}\cdot v = 1.$$

Furthermore,

$$\mathbf{r}''(t) = \frac{1}{v^2}\underline{\mathbf{e}}\begin{pmatrix} -a\cos(t/v) \\ -a\sin(t/v) \\ 0 \end{pmatrix}$$

and so the curvature

$$\kappa \overset{\text{def}}{=} |\mathbf{r}''(t)| = \frac{a}{v^2} = \frac{a}{a^2+b^2}.$$

As $a \to 0^+$ while b is constant, $\kappa \to 0$, as one would expect, since the curve is 'straightened' like a stubborn cable. Also notice that $\kappa \to 1/a$ as $b \to 0$, also as one would expect, because in this case the helix tends to a circle of radius a.

Notice that all examples so far have involved curves of constant curvature. We will investigate more complicated curves after we have found a simpler method of determining the curvature. After all, it is rather cumbersome to determine a unit-speed parameterisation every time.

Proposition NN

Let $\mathbf{r}\colon I \to \mathbb{R}^3$ be any regular f-curve (unit-speed or not!). Then

$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$$

where a dot denotes a derivative w.r.t. the parameter and the $I \ni t$-dependance is understood.

Proof

Let $s \mapsto \mathbf{q}(s)$ be a unit-speed reparameterisation of the given f-curve $t \mapsto \mathbf{r}(t)$. Then

$$\mathbf{q}\big(s(t)\big) = \mathbf{r}(t).$$

Differentiation w.r.t. t yields

$$\frac{d\mathbf{q}}{ds} \cdot \frac{ds}{dt} = \frac{d\mathbf{r}}{dt}$$

and

$$\frac{d^2\mathbf{q}}{ds^2} \cdot \left(\frac{ds}{dt}\right)^2 + \frac{d\mathbf{q}}{ds} \cdot \frac{d^2s}{dt^2} = \frac{d^2\mathbf{r}}{dt^2}.$$

Thus

$$\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} = \left(\frac{d\mathbf{q}}{ds} \cdot \frac{ds}{dt}\right) \times \left(\frac{d^2\mathbf{q}}{ds^2} \cdot \left(\frac{ds}{dt}\right)^2 + \frac{d\mathbf{q}}{ds} \cdot \frac{d^2s}{dt^2}\right) = \frac{d\mathbf{q}}{ds} \cdot \frac{ds}{dt} \times \frac{d^2\mathbf{q}}{ds^2} \cdot \left(\frac{ds}{dt}\right)^2$$

and so

$$\left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right| = \left|\frac{ds}{dt} \cdot \left(\frac{ds}{dt}\right)^2\right| \kappa$$

since $\mathbf{q}$ is unit-speed and by the definition $\kappa \overset{\text{def}}{=} \left|\frac{d^2\mathbf{q}}{ds^2}\right|$. (↑) also yields

$$\left|\frac{d\mathbf{r}}{dt}\right| = \left|\frac{ds}{dt}\right|;$$

thus

$$\frac{\left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right|}{\left|\frac{d\mathbf{r}}{dt}\right|^3} = \frac{\left|\frac{ds}{dt} \cdot \left(\frac{ds}{dt}\right)^2\right| \kappa}{\left|\frac{ds}{dt}\right|^3} = \kappa.$$

■

It might appears as if Proposition NN only holds for plane curves. However, since any plane curve can be trivially embedded in $\mathbb{R}^3$ by means of the linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix},$$

we can use Proposition NN even for planar curves. Now we can investigate some more complicated curves with less effort. If we are interested in curves with non-constant curvature, it is natural to turn to the spirals of Example NN.

Example NN

We will compute the curvature of the Archimedean spiral, which is parameterised by

$$\mathbf{r}(t) = \frac{1}{5}\underline{\mathbf{e}}\begin{pmatrix} t\cos t \\ t\sin t \\ 0 \end{pmatrix}, \qquad t > 0.$$

Thus

$$\mathbf{r}'(t) = \frac{1}{5}\underline{\mathbf{e}}\begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \\ 0 \end{pmatrix}, \qquad \mathbf{r}''(t) = \frac{1}{5}\underline{\mathbf{e}}\begin{pmatrix} -2\sin t - t\cos t \\ 2\cos t - t\sin t \\ 0 \end{pmatrix},$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \frac{1}{25}\underline{\mathbf{e}}\begin{pmatrix} 0 \\ 0 \\ 2+t^2 \end{pmatrix}, \qquad |\mathbf{r}'(t)\times\mathbf{r}''(t)| = \frac{1}{25}(2+t^2),$$

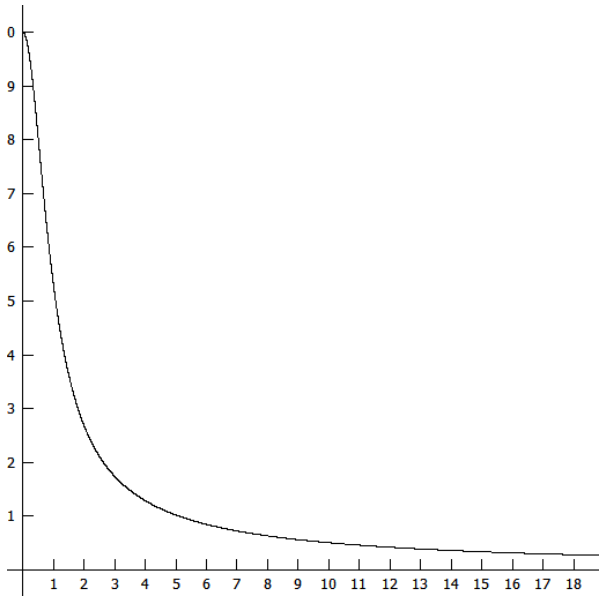
and

$$|\mathbf{r}'(t)| = \frac{1}{5}\sqrt{1+t^2}.$$

Therefore,

$$\kappa = \frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = 5\frac{(2+t^2)}{(1+t^2)^{3/2}}.$$

It is easy to show that $d\kappa/dt < 0, \forall t > 0$. Below is the graph of $t \mapsto \kappa(t)$.



The logarithmic spiral is left as an exercise.

We end this subsection by investigating if there is a curve in which the curvature is proportional to arc length. (Intuitively, the existence part is rather obvious. But it is far from obvious if there is a reasonably simple parameterisation for it.)

Let $\gamma = \mathbf{r}(I)$ where $\mathbf{r}\colon I \to \mathbb{R}^2$ is a unit-speed parameterisation. Our requirement is thus

$$\kappa \overset{\text{def}}{=} |\mathbf{r}''(t)| = at, \qquad t > 0,$$

where $a > 0$. Since $\mathbf{r}$ is unit-speed, $|\mathbf{r}'(t)| = 1, \forall t \in I$, and, since *any* vector of unit length may be written $(\cos\varphi\,, \sin\varphi)$ for some $\varphi \in \mathbb{R}$, it is clear that there must exist a function $t \mapsto \varphi(t)$ such that

$$\mathbf{r}'(t) = (\cos\varphi(t)\,, \sin\varphi(t)).$$

Differentiation yields

$$\mathbf{r}''(t) = \varphi'(t)(-\sin\varphi(t)\,, \cos\varphi(t))$$

and so

$$|\mathbf{r}''(t)| = |\varphi'(t)| = at.$$

If we require that $t \mapsto \varphi$ is an increasing function,

$$\varphi'(t) = at$$

which yields, if we require $\varphi(0) = 0$ [curve is horizontal at $t = 0$],

$$\varphi(t) = \frac{1}{2}at^2.$$

Thus,

$$\mathbf{r}'(t) = \left(\cos\frac{1}{2}at^2\,, \sin\frac{1}{2}at^2\right)$$

and so, if position our curve on the plane in such a way that $\mathbf{r}(0) = \mathbf{0}$,

$$\mathbf{r}(t) = \left(\sqrt{\frac{2}{a}}C\left(\sqrt{\frac{a}{2}}t\right), \sqrt{\frac{2}{a}}S\left(\sqrt{\frac{a}{2}}t\right)\right)$$

where

$$C(x) \coloneqq \int_0^x \cos t^2\, dt\,, \qquad S(x) \coloneqq \int_0^x \sin t^2\, dt$$

are the *Fresnel integrals*, which are well-known non-elementary functions. For the obvious reason, the curve $\mathbf{r}(t)$, which is called the *Euler spiral* (or *Cornu spiral*), is often used in railway construction. Below the image $\mathbf{r}(I)$ is shown (where $I = [-100, 100]$) for $a = 2$. Although we initially required $t > 0$, $\mathbf{r}(t)$ certainly makes sense for all $t \in \mathbb{R}$, and $\mathbf{r}(-t) = -\mathbf{r}(t), \forall t \in \mathbb{R}$. The definition equation (↑) still holds with a minor modification: $\kappa \overset{\text{def}}{=} |\mathbf{r}''(t)| = a|t|,\ t \in \mathbb{R}$.

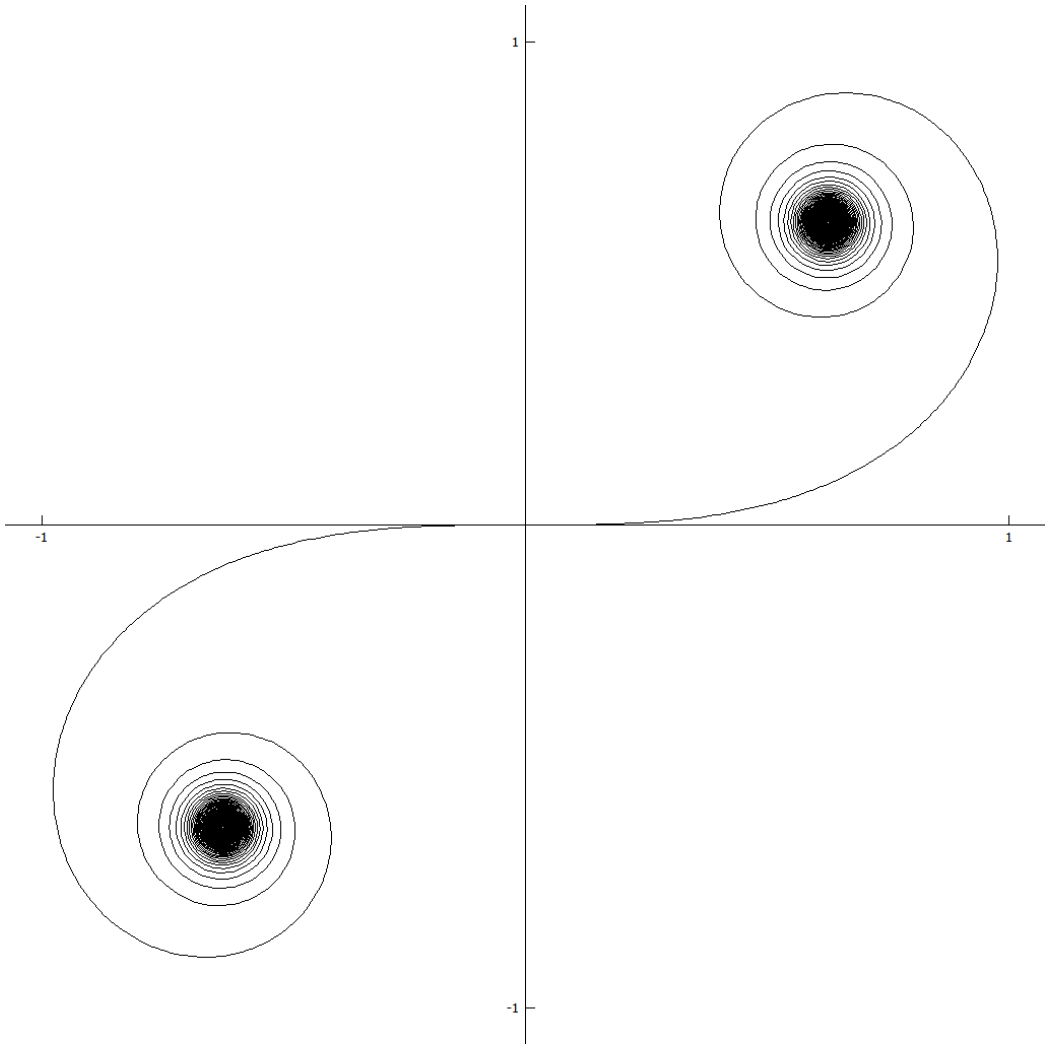


Figure 36. An Euler Spiral.

4.2.6 Torsion

It is intuitively clear, and one can rigorously show that, up to an isometry of the plane, the curvature function $t \mapsto \kappa$ uniquely determines a plane curve. However, it is evident that the same does not apply for space curves. Indeed, a (circular cylindrical) helix with parameters $a = b = 1$ has the same curvature as a circle with radius $a = 2$, namely, $1/2$, and there is no isometry of space that can make a helix out of a circle or vice versa! In this section, we will introduce a new function of t, besides the curvature, namely, *torsion*. Together, the curvature and torsion uniquely determine a space curve up to an isometry of space. Essentially, we will see that the torsion measures the failure of a space curve to lie inside a single plane in space. First, we need some 'new' concepts.

Definition NN

Let $\gamma = \mathbf{r}(I)$ be a space curve with $\mathbf{r}$ regular and unit-speed. Then, at $t \in I$, $\hat{\mathbf{t}}(t) \coloneqq \mathbf{r}'(t)$ is called the *unit tangent (vector)*, and, assuming $\kappa(t) \neq 0$, $\hat{\mathbf{n}}(t) \coloneqq \frac{1}{\kappa}\mathbf{r}''(t)\left(= \frac{1}{|\mathbf{r}''(t)|}\mathbf{r}''(t)\right)$, is called the *unit normal (vector)* to γ.

Recall from Lemma NN that $\hat{\mathbf{t}}(t) \perp \hat{\mathbf{n}}(t)$ at every point $t \in I$ where the unit normal is defined. It should be clear that the unit tangent and normal vectors depend only on the curve γ *and its orientation*, and not on the parameterisation otherwise.

Definition NN

Let $\gamma = \mathbf{r}(I)$ be a space curve with $\mathbf{r}$ regular and unit-speed. Then, at every $t \in I$ where the unit normal $\hat{\mathbf{n}}(t)$ is defined,

$$\hat{\mathbf{b}}(t) \coloneqq \hat{\mathbf{t}}(t) \times \hat{\mathbf{n}}(t)$$

is called the *binormal* to γ.

Since $\hat{\mathbf{t}}(t)$ and $\hat{\mathbf{n}}(t)$ are perpendicular unit vectors, $\hat{\mathbf{b}}(t)$ is too of unit length; HTH (hence the hat).

Consider now a regular, unit-speed planar space curve $\gamma = \mathbf{r}(I)$ with unit tangent and normal vectors $\hat{\mathbf{t}}(t)$ and $\hat{\mathbf{n}}(t)$ at $t \in I$, with I open, respectively, and let $\Pi \subset \mathbb{R}^3$ be a plane such that $\gamma \cap \Pi = \gamma$. Let $\Pi_0 \subset \mathbb{R}^3$ be a plane with the same normal direction as Π but translated, if necessary, as to contain the origin; notice that Π_0 is a vector space. Since I is open, there exists, for every $t \in I$ a sufficiently small $\epsilon > 0$ such that $t + \epsilon \in I$, too. By definition,

$$\hat{\mathbf{t}}(t) = \mathbf{r}'(t) = \lim_{\epsilon \to 0^+} \frac{\mathbf{r}(t+\epsilon) - \mathbf{r}(t)}{\epsilon}.$$

Since both $\mathbf{r}(t+\epsilon) \in \Pi$ and $\mathbf{r}(t) \in \Pi$, it follows that $\mathbf{r}(t+\epsilon) - \mathbf{r}(t) \in \Pi_0$. This being so for every positive ϵ in some neighbourhood of 0, it follows that the limit $\hat{\mathbf{t}}(t) \in \Pi_0$ as well. Hence,

$$\hat{\mathbf{t}}(t) \in \Pi_0, \qquad \forall t \in I.$$

The unit normal, if defined, is

$$\hat{\mathbf{n}}(t) = \frac{1}{\kappa}\hat{\mathbf{t}}'(t) = \frac{1}{\kappa}\lim_{\epsilon \to 0^+} \frac{\hat{\mathbf{t}}(t+\epsilon) - \hat{\mathbf{t}}(t)}{\epsilon}.$$

But Π_0 is a vector space, and so (↑) implies $\hat{\mathbf{t}}(t+\epsilon) - \hat{\mathbf{t}}(t) \in \Pi_0$ for every $\epsilon > 0$ in a neighbourhood of 0; thus

$$\hat{\mathbf{n}}(t) \in \Pi_0, \qquad \forall t \in I \text{ where } \hat{\mathbf{n}}(t) \text{ is defined.}$$

Since $\hat{\mathbf{t}}(t)$ and $\hat{\mathbf{n}}(t)$ are perpendicular unit vectors in Π_0, the binormal $\hat{\mathbf{b}}(t) \overset{\text{def}}{=} \hat{\mathbf{t}}(t) \times \hat{\mathbf{n}}(t) \perp \Pi_0$ for all t. Thus, not only the (unit) length, but also *the direction of* $\hat{\mathbf{b}}(t)$ *is constant*. We summarise this as a proposition:

Proposition NN

Let $\gamma = \mathbf{r}(I)$ be a planar space curve ($\mathbf{r}$ regular and unit-speed) with binormal $\hat{\mathbf{b}}(t)$ at $t \in I$ (where defined). Then

$$\left(\hat{\mathbf{b}}\right)'(t) = 0, \qquad \forall t \in I$$

(where defined).

It is therefore tempting to define the torsion as being closely related to the magnitude of the derivative of the binormal vector, and we will do so. Notice that

$$\hat{\mathbf{b}} \overset{\text{def}}{=} \hat{\mathbf{t}} \times \hat{\mathbf{n}} \Rightarrow \left(\hat{\mathbf{b}}\right)' = (\hat{\mathbf{t}})' \times \hat{\mathbf{n}} + \hat{\mathbf{t}} \times (\hat{\mathbf{n}})' = \hat{\mathbf{t}} \times (\hat{\mathbf{n}})';$$

thus $\left(\hat{\mathbf{b}}\right)' \perp \hat{\mathbf{t}}$. In addition, since $\left|\hat{\mathbf{b}}\right| \equiv 1$, $\left(\hat{\mathbf{b}}\right)' \perp \hat{\mathbf{b}}$. But since $\hat{\mathbf{b}} \overset{\text{def}}{=} \hat{\mathbf{t}} \times \hat{\mathbf{n}}$, we have $\left(\hat{\mathbf{b}}\right)' \parallel \hat{\mathbf{n}}$.

Definition NN

Let $\gamma = \mathbf{r}(I)$ be a space curve with $\mathbf{r}$ regular and unit-speed. Then the *torsion* τ at $t \in I$ is given by the relation

$$\left(\hat{\mathbf{b}}\right)' = -\tau\hat{\mathbf{n}}$$

(where defined).

The minus sign is a convention. Notice that $\hat{\mathbf{n}}$ (and so $\hat{\mathbf{b}}$) is only defined where $\kappa \neq 0$. This means that τ is also only defined on (open) intervals where $\kappa \neq 0$. Notice also that $\tau = -\left(\hat{\mathbf{b}}\right)' \cdot \hat{\mathbf{n}}$ and $|\tau| = \left|\left(\hat{\mathbf{b}}\right)'\right|$.

Corollary NN

A regular *planar* space curve has zero torsion everywhere it is defined.

Naturally, we wish to compute the torsion of a curve, without all-embracing trouble. In other words, we need an analogue of Proposition NN, and here it comes:

Proposition NN

Let $\gamma = \mathbf{r}(I)$ be a space curve with $\mathbf{r}$ regular (but not necessarily unit-speed). Then

$$\tau = \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \dddot{\mathbf{r}}}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2}$$

where τ is defined.

Proof

Following the proof of Proposition NN, we introduce the unit-speed reparameterisation function $s \mapsto \mathbf{q}$ and so, simply copying the results obtained there,

$$\mathbf{r}(t) = \mathbf{q}\big(s(t)\big), \qquad \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{q}}{ds} \cdot \frac{ds}{dt}, \qquad \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{q}}{ds^2} \cdot \left(\frac{ds}{dt}\right)^2 + \frac{d\mathbf{q}}{ds} \cdot \frac{d^2s}{dt^2}$$

and

$$\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{q}}{ds} \cdot \frac{ds}{dt} \times \frac{d^2\mathbf{q}}{ds^2} \cdot \left(\frac{ds}{dt}\right)^2, \qquad \left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right| = \left|\frac{ds}{dt} \cdot \left(\frac{ds}{dt}\right)^2\right| \kappa.$$

We now also need

$$\frac{d^3\mathbf{r}}{dt^3} = \frac{d^3\mathbf{q}}{ds^3} \cdot \left(\frac{ds}{dt}\right)^3 + \frac{d^2\mathbf{q}}{ds^2} \cdot 2\left(\frac{ds}{dt}\right) \cdot \frac{d^2s}{dt^2} + \frac{d^2\mathbf{q}}{ds^2} \cdot \frac{ds}{dt} \cdot \frac{d^2s}{dt^2} + \frac{d\mathbf{q}}{ds} \cdot \frac{d^3s}{dt^3}.$$

Putting it all together,

$$\frac{(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\cdot\dddot{\mathbf{r}}}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2}=\frac{1}{\left|\frac{ds}{dt}\cdot\left(\frac{ds}{dt}\right)^2\right|^2\kappa^2}\cdot\left(\frac{d\mathbf{q}}{ds}\cdot\frac{ds}{dt}\times\frac{d^2\mathbf{q}}{ds^2}\cdot\left(\frac{ds}{dt}\right)^2\right)\cdot$$
$$\cdot\left(\frac{d^3\mathbf{q}}{ds^3}\cdot\left(\frac{ds}{dt}\right)^3+\frac{d^2\mathbf{q}}{ds^2}\cdot 2\left(\frac{ds}{dt}\right)\cdot\frac{d^2s}{dt^2}+\frac{d^2\mathbf{q}}{ds^2}\cdot\frac{ds}{dt}\cdot\frac{d^2s}{dt^2}+\frac{d\mathbf{q}}{ds}\cdot\frac{d^3s}{dt^3}\right)=$$
$$=\frac{1}{\left(\frac{ds}{dt}\right)^3\kappa^2}\cdot\left(\frac{d\mathbf{q}}{ds}\times\frac{d^2\mathbf{q}}{ds^2}\right)\cdot\left(\frac{d^3\mathbf{q}}{ds^3}\left(\frac{ds}{dt}\right)^3+3\frac{d^2\mathbf{q}}{ds^2}\frac{ds}{dt}\frac{d^2s}{dt^2}+\frac{d\mathbf{q}}{ds}\frac{d^3s}{dt^3}\right)=$$
$$=\frac{1}{\left(\frac{ds}{dt}\right)^3\kappa^2}\cdot(\hat{\mathbf{t}}\times\kappa\hat{\mathbf{n}})\cdot\left(\frac{d}{ds}(\kappa\hat{\mathbf{n}})\left(\frac{ds}{dt}\right)^3+3\kappa\hat{\mathbf{n}}\frac{ds}{dt}\frac{d^2s}{dt^2}+\hat{\mathbf{t}}\frac{d^3s}{dt^3}\right)=$$
$$=\frac{1}{\left(\frac{ds}{dt}\right)^3\kappa^2}\cdot\left(\kappa\hat{\mathbf{b}}\right)\cdot\left(\frac{d}{ds}(\kappa\hat{\mathbf{n}})\left(\frac{ds}{dt}\right)^3+3\kappa\hat{\mathbf{n}}\frac{ds}{dt}\frac{d^2s}{dt^2}+\hat{\mathbf{t}}\frac{d^3s}{dt^3}\right)=$$
$$=\frac{1}{\left(\frac{ds}{dt}\right)^3\kappa^2}\cdot\left(\kappa\hat{\mathbf{b}}\right)\cdot\left(\frac{d}{ds}(\kappa\hat{\mathbf{n}})\left(\frac{ds}{dt}\right)^3\right)=\frac{1}{\kappa}\hat{\mathbf{b}}\cdot\left(\frac{d}{ds}(\kappa\hat{\mathbf{n}})\right)=\frac{1}{\kappa}\hat{\mathbf{b}}\cdot\left(\frac{d\kappa}{ds}\hat{\mathbf{n}}+\kappa\frac{d\hat{\mathbf{n}}}{ds}\right)=$$
$$=\hat{\mathbf{b}}\cdot\frac{d\hat{\mathbf{n}}}{ds}.$$

Since

$$0=\frac{d}{ds}(0)=\frac{d}{ds}\left(\hat{\mathbf{b}}\cdot\hat{\mathbf{n}}\right)=\frac{d\hat{\mathbf{b}}}{ds}\cdot\hat{\mathbf{n}}+\hat{\mathbf{b}}\cdot\frac{d\hat{\mathbf{n}}}{ds}=-\tau+\hat{\mathbf{b}}\cdot\frac{d\hat{\mathbf{n}}}{ds}$$

we have

$$\frac{(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\cdot\dddot{\mathbf{r}}}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2}=\hat{\mathbf{b}}\cdot\frac{d\hat{\mathbf{n}}}{ds}=\tau$$

as promised. ■

Example NN

Let us compute the torsion of the (circular cylindrical) helix, given by

$$\mathbf{r}(t)=\underline{\mathbf{e}}\begin{pmatrix}a\cos t\\ a\sin t\\ bt\end{pmatrix},\qquad t\in\mathbb{R}.$$

We find

$$\mathbf{r}'(t)=\underline{\mathbf{e}}\begin{pmatrix}-a\sin t\\ a\cos t\\ b\end{pmatrix},\qquad \mathbf{r}''(t)=\underline{\mathbf{e}}\begin{pmatrix}-a\cos t\\ -a\sin t\\ 0\end{pmatrix},\qquad \mathbf{r}'''(t)=\underline{\mathbf{e}}\begin{pmatrix}a\sin t\\ -a\cos t\\ 0\end{pmatrix},$$
$$\mathbf{r}'(t)\times\mathbf{r}''(t)=\underline{\mathbf{e}}\begin{pmatrix}ab\sin t\\ -ab\cos t\\ a^2\end{pmatrix},\qquad |\mathbf{r}'(t)\times\mathbf{r}''(t)|=\sqrt{a^2b^2+a^4}=a\sqrt{a^2+b^2}.$$

Thus

$$\frac{\big(\mathbf{r}'(t)\times\mathbf{r}''(t)\big)\cdot\mathbf{r}'''(t)}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|^2}=\frac{a^2b}{a^2(a^2+b^2)}=\frac{b}{a^2+b^2}.$$

Notice that $\tau\to 0$ as $b\to 0$, which is to be expected, for in this limit, the helix becomes a planar circle.

4.2.7 The Frenet—Serret Formulae

The formulae relating the unit tangent, normal, and binormal vectors to the curvature and torsion of a space curve are rather succinctly and neatly summarized in the three Frenet—Serret formulae, two of which we already know:

$$\frac{d\hat{\mathbf{t}}}{ds}=\kappa\hat{\mathbf{n}},\qquad \frac{d\hat{\mathbf{b}}}{ds}=-\tau\hat{\mathbf{n}}$$

where s is an arc-length parameter. The last Frenet—Serret formula follows from a simple calculation:

$$\hat{\mathbf{b}}\overset{\text{def}}{=}\hat{\mathbf{t}}\times\hat{\mathbf{n}}\Rightarrow\hat{\mathbf{n}}=\hat{\mathbf{b}}\times\hat{\mathbf{t}}\Rightarrow\frac{d\hat{\mathbf{n}}}{ds}=\frac{d\hat{\mathbf{b}}}{ds}\times\hat{\mathbf{t}}+\hat{\mathbf{b}}\times\frac{d\hat{\mathbf{t}}}{ds}=(-\tau\hat{\mathbf{n}})\times\hat{\mathbf{t}}+\hat{\mathbf{b}}\times(\kappa\hat{\mathbf{n}})=\tau\hat{\mathbf{b}}-\kappa\hat{\mathbf{t}},$$

that is,

$$\frac{d\hat{\mathbf{n}}}{ds}=\tau\hat{\mathbf{b}}-\kappa\hat{\mathbf{t}}.$$

The beauty of the Frenet—Serret formulae lies in their matrix form:

Proposition NN (The Frenet—Serret Theorem)

Let a regular space curve γ have unit tangent, normal, and binormal vectors $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{b}}$, respectively, and let the curvature and torsion be κ and τ, respectively, and let s be an arc-length parameter, a derivative relative to which is denoted by a prime. Then

$$\begin{pmatrix}(\hat{\mathbf{t}})'\\(\hat{\mathbf{n}})'\\(\hat{\mathbf{b}})'\end{pmatrix}=\begin{pmatrix}0&\kappa&0\\-\kappa&0&\tau\\0&-\tau&0\end{pmatrix}\begin{pmatrix}\hat{\mathbf{t}}\\\hat{\mathbf{n}}\\\hat{\mathbf{b}}\end{pmatrix}.$$

4.2.8 A Modern Definition of a Curve

Our definition of a (parameterised) curve is a simple and highly intuitive one. However, it has some drawbacks. Perhaps most importantly, it will differ rather significantly from the definition of a surface that we will give in the next section. To remedy this, we give an alternative, and more 'modern', definition of a curve.

Definition NN

A subset $\gamma\subset\mathbb{R}^n$ is called a *(manifold) curve* iff, for every point $\mathbf{x}\in\gamma$, there exists an open set $U\subset\mathbb{R}^n$ containing $\mathbf{x}$ and an open set $V\in\mathbb{R}$ such that $U\cap\gamma$ and V are homeomorphic.

This definition is not equivalent to the definition of a (*parameterised) curve* given as Definition NN. Still, in many ways, it is a more 'natural' definition. It captures the essential property of a curve, namely, that a curve, *locally* 'looks like' a small, deformed part of an interval of $\mathbb{R}$. In addition, it states that, locally, about every point, one can introduce a parameterisation of the curve,

and this is a very nice f-curve, indeed a homeomorphism. Also, one does not have to display an f-curve, the image of which is γ, in order to show that γ is a (manifold) curve. It suffices to show that, locally, it can be parameterised by f-curves.

One of the main drawbacks of this definition is that it does not allow non-simple curves. This is also the reason why we have used a different definition in this section.

4.3 Surfaces

We now turn to two-dimensional surfaces embedded in $\mathbb{R}^3$. As for the definition of 'surface', we could certainly mimic the definition of a curve made in the last section:

Definition NN

A *(parameterised) surface* in $\mathbb{R}^3$ is the image $\Sigma = \mathbf{r}(D)$ of a *surface parameterisation function* (or an *f-surface* for short) $\mathbf{r}\colon D \to \mathbb{R}^3$ where $\mathbf{r}$ is injective and $D \subset \mathbb{R}^2$ is a simply connected set.

A surface parameterisation function is *regular* iff it is smooth and $|\mathbf{r}_x \times \mathbf{r}_y| \neq 0$ at every point. A surface is regular iff it is the image of some regular surface parameterisation function.

Notice that this definition disqualifies surfaces with self-intersections, but this is a rather small problem: while we are often interested in curves that intersect themselves, we rarely need to consider surfaces with this property. In addition, should such a need emerge, you are likely to overcome the problem by subdividing the parameter region D into smaller regions, such that the restrictions of the surface parameterisation function to these small regions are injective.

Although this definition works perfectly for the vast majority of every-day situations, there are a few problems with it. One is that, in order to show that a subset of $\mathbb{R}^3$ is a surface, we need to display a parameterisation function. Thus, since we are not interested in self-intersecting surfaces anyway, there is really no reason *not* to go for the surface analogue of the 'modern' definition of a curve.

Indeed, it is clear that if $\mathbf{x} \in S^2$ is any point on the unit sphere, then there is an open set $U \subset \mathbb{R}^3$ containing $\mathbf{x}$ and an open subset $V \subset \mathbb{R}^2$ such that $U \cap S^2$ and V are homeomorphic, and this is typical for a nice surface in space. We will therefore make

Definition NN

A subset $\Sigma \subset \mathbb{R}^n$ is called a *(manifold) surface* iff, for every point $\mathbf{x} \in \Sigma$, there exists an open set $U \subset \mathbb{R}^n$ containing $\mathbf{x}$ and an open set $V \subset \mathbb{R}^2$ such that $U \cap \Sigma$ and V are homeomorphic.

A pair (V, Φ) of a non-empty open subset $V \subset \mathbb{R}^2$ and a homeomorphism $\Phi\colon V \to \Sigma \cap U$ for some open $U \subset \mathbb{R}^3$ is called a *local coordinate system*, or a *(local coordinate) patch (or chart)* of Σ. A collection of coordinate patches (V_i, Φ_i) is called an *atlas* for Σ if $\bigcup \Phi(V_i) = \Sigma$.

It follows that every surface has an atlas, and if a subset $\Sigma \subset \mathbb{R}^3$ has an atlas, then it is a surface. In the following examples, we will mostly study surfaces with an atlas consisting of a single chart. Thus, in these simple cases any of the definitions would do.

4.3.1 Examples of Surfaces

Example NN

The *unit sphere* S^2 is a manifold surface. Indeed, consider the functions

$$\mathbf{r}_1(\theta, \varphi) = \underline{\mathbf{e}} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}, \qquad (\theta, \varphi) \in V_1 :=]0, \pi[\times]0, 2\pi[,$$

$$\mathbf{r}_2(\theta,\varphi) = \underline{\mathbf{e}}\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}, \qquad (\theta,\varphi)\in V_2 \coloneqq]0,\pi[\times]-\pi,\pi[,$$

$$\mathbf{r}_3(x,y) = \underline{\mathbf{e}}\begin{pmatrix}x\\ y\\ \sqrt{1-x^2-y^2}\end{pmatrix}, \qquad (x,y)\in V_3 \coloneqq \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2<\frac{1}{10}\right\},$$

$$\mathbf{r}_4(x,y) = \underline{\mathbf{e}}\begin{pmatrix}x\\ y\\ -\sqrt{1-x^2-y^2}\end{pmatrix}, \qquad (x,y)\in V_4 \coloneqq \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2<\frac{1}{10}\right\}.$$

It is clear that every pair $(V_i,\mathbf{r}_i)$ is a local coordinate patch, and that $\cup\,\mathbf{r}_i(V_i) = S^2$. Thus the collection $(V_i,\mathbf{r}_i)$ of patches constitute an atlas for S^2, which therefore is a manifold surface.

Example NN

A *plane* is the image of

$$\mathbf{r}(u,v) = \mathbf{p}_0 + u\mathbf{u} + v\mathbf{v}, \qquad (u,v)\in\mathbb{R}^2$$

where $\mathbf{p}_0\in\mathbb{R}^3$ is some fixed point and $\mathbf{u}$ and $\mathbf{v}$ are two linearly independent vectors.

Example NN

A circular cylinder of radius $a>0$, given by the Cartesian equation $x^2+y^2=a^2$, is the image of

$$\mathbf{r}(\varphi,z) = \underline{\mathbf{e}}\begin{pmatrix}a\cos\varphi\\ a\sin\varphi\\ z\end{pmatrix}, \qquad (\varphi,z)\in[0,2\pi[\times\mathbb{R}.$$

Although $\mathbf{r}$ does not qualify for a chart, it is clear that the circular cylinder is a surface. For instance, an atlas is given by the two charts

$$\mathbf{r}_1(\varphi,z) = \underline{\mathbf{e}}\begin{pmatrix}a\cos\varphi\\ a\sin\varphi\\ z\end{pmatrix}, \qquad (\varphi,z)\in V_1 \coloneqq]0,2\pi[\times\mathbb{R},$$

$$\mathbf{r}_2(\varphi,z) = \underline{\mathbf{e}}\begin{pmatrix}a\cos\varphi\\ a\sin\varphi\\ z\end{pmatrix}, \qquad (\varphi,z)\in V_2 \coloneqq]-\pi,\pi[\times\mathbb{R}.$$

Example NN

The set of points that satisfy $x^2+y^2=z^2$ is an example of a (circular) *cone* and is the image of

$$\mathbf{r}(\varphi,z) = \underline{\mathbf{e}}\begin{pmatrix}z\cos\varphi\\ z\sin\varphi\\ z\end{pmatrix}, \qquad (\varphi,z)\in[0,2\pi[\times\mathbb{R}.$$

The cone is *not* a manifold surface, because it does not 'look like' a Euclidean plane at the vertex $(0,0,0)$. However, the restriction $z>0$ (or $z<0$) *is* a manifold surface.

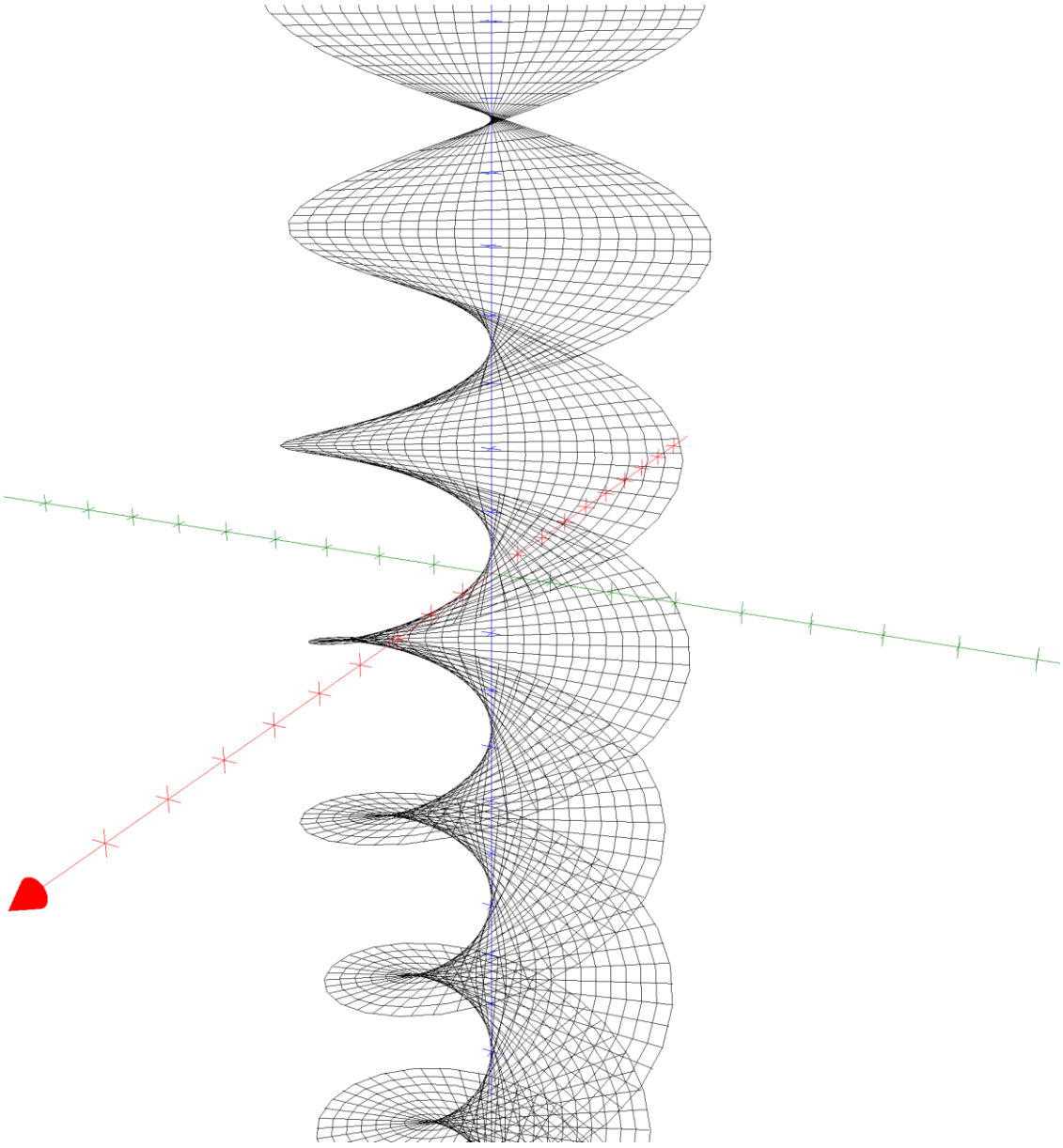
Example NN

The *helicoid* is the image of

$$\mathbf{r}(u,v) = \underline{\mathbf{e}}\begin{pmatrix} a\,u\cos v \\ a\,u\sin v \\ bv \end{pmatrix}, \qquad (u,v) \in D$$

for some $a, b > 0$ and some open $D \subset \mathbb{R}^2$.

It is apparent from the parameterisation that a helicoid is the surface traced out by a rotating (straight-line) aircraft or boat 'propeller' as the vehicle is moving with constant velocity $\mathbf{v} = b\hat{\mathbf{z}}$ in which case v is the time. Below a helicoid is drawn for $(a,b) = (2,1)$ and $D = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\times \left]-6\pi, 6\pi\right[$.



Example NN

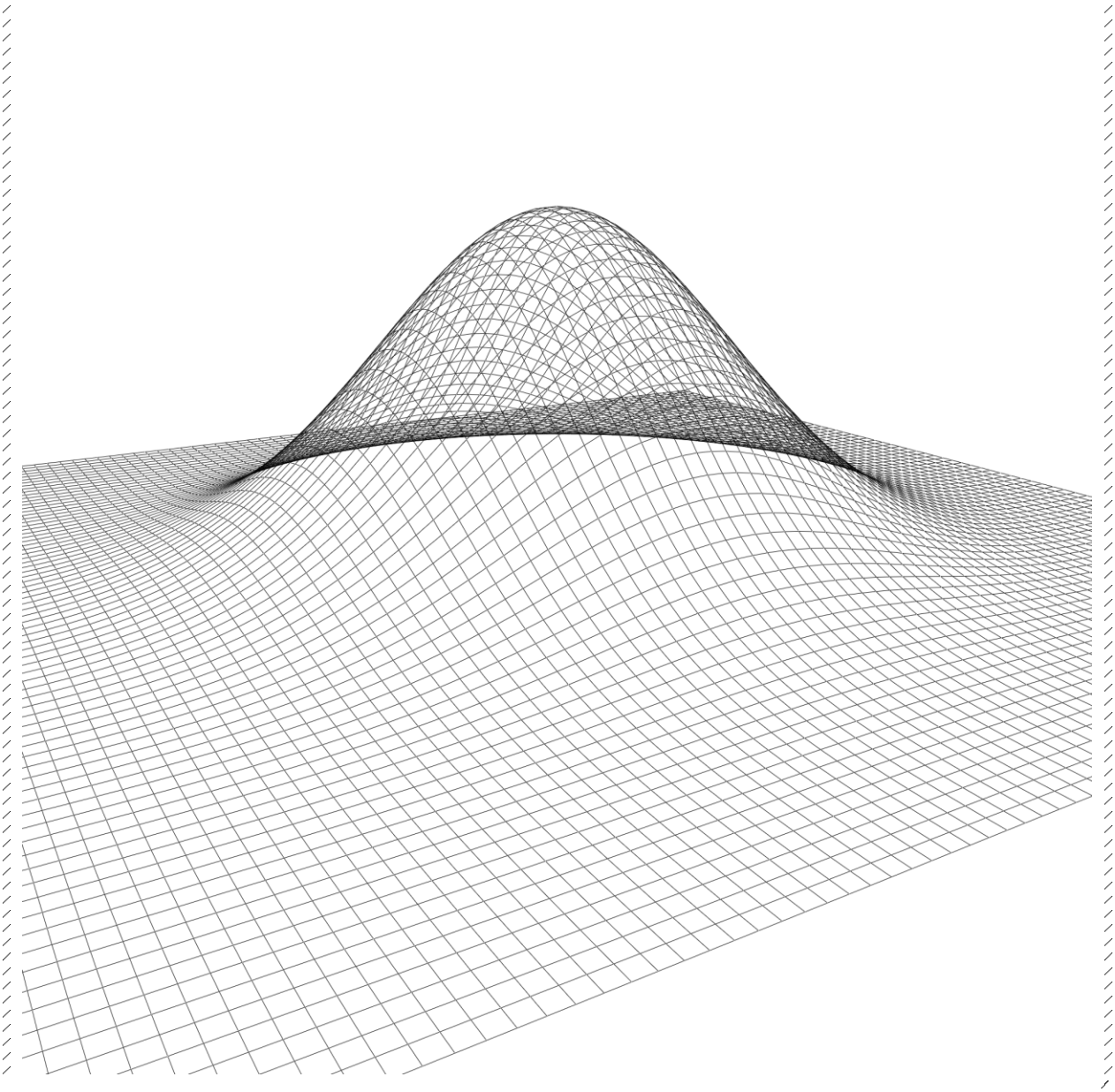
Let $f: D \to \mathbb{R}$ be a real valued function of two variables defined on $D \subseteq \mathbb{R}^2$. If f is continuous and D open, then the *graph*, defined by

$$\{(x, y, z) \in \mathbb{R}^3 : ((x, y) \in D) \wedge (z = f(x, y))\}$$

is a surface, parameterised by the single patch

$$\mathbf{r}(x, y) = \underline{\mathbf{e}} \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix}, \qquad (x, y) \in D.$$

Below is the graph of the two-dimensional Gaussian $f(x, y) = 5e^{-\frac{1}{8}(x^2+y^2)}$ on $(x, y) \in]-10, 10[^2$.



4.3.2 The Tangent Space and the Standard Normal

Consider a surface $\Sigma = \mathbf{r}(D)$ with coordinates $(u, v) \in D$. At every point $(u, v) \in D$, mapped to $\mathbf{x} \coloneqq \mathbf{r}(u, v) \in \Sigma$, the vectors $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$ are tangent to the surface. The plane

$$\Pi_{\Sigma,\mathbf{x}} \coloneqq \{\mathbf{q} \in \mathbb{R}^3 \colon \mathbf{q} = \mathbf{r}(u, v) + s\mathbf{r}_u(u, v) + t\mathbf{r}_v(u, v), (s, t) \in \mathbb{R}^2\}$$

is called the *tangent plane* of Σ at $\mathbf{r}(u, v)$. Although $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$ depend upon the parameterisation function $\mathbf{r}$ of Σ, the tangent plane $\Pi_{\Sigma,\mathbf{x}}$ does not. Notice also that $\Pi_{\Sigma,\mathbf{x}}$ *is* a plane in space, because $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$ are guaranteed to be non-parallel since $\mathbf{r}$ is regular. The tangent plane $\Pi_{\Sigma,\mathbf{x}}$ is not a vector subspace of $\mathbb{R}^3$ unless $\mathbf{0} \in \Pi_{\Sigma,\mathbf{x}}$. Since being a vector space is a formal nicety, we introduce the *tangent space* $T\Sigma_{\mathbf{x}}$ as the tangent plane translated in $\mathbb{R}^3$ as to contain the origin. Thus, the tangent space $T\Sigma_{\mathbf{x}}$ *is* a vector space, and a plane with the same normal direction as $\Pi_{\Sigma,\mathbf{x}}$. Of course, if you like, you can imagine $T\Sigma_{\mathbf{x}}$ as being a vector subspace of a copy of $\mathbb{R}^3$ with origin at $\mathbf{x}$ relative to the 'underlying' $\mathbb{R}^3$, in which Σ lives.

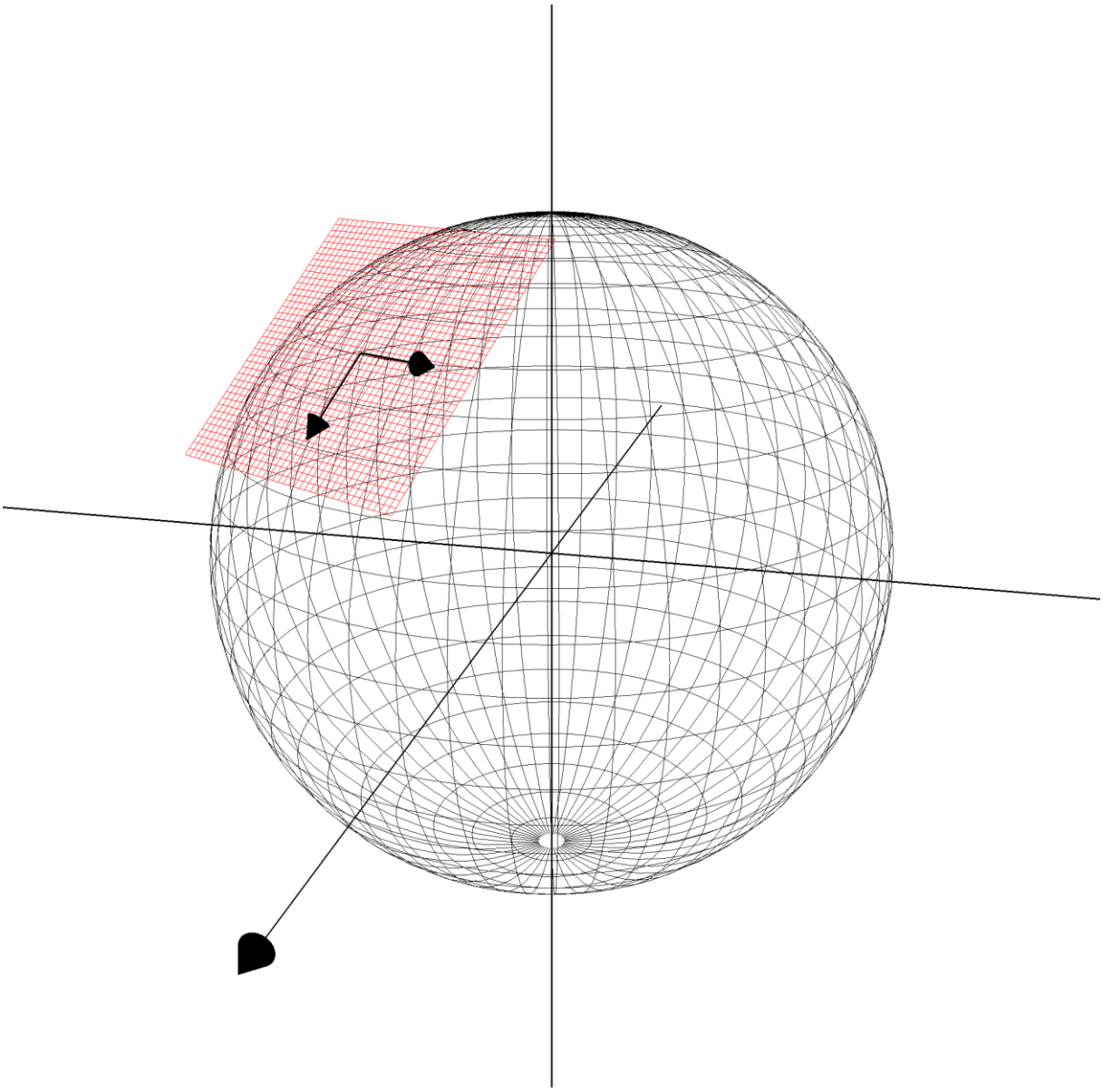


Figure 37. The tangent space at $(\theta, \varphi) = \left(\frac{\pi}{4}, 0.7 \cdot \frac{\pi}{2}\right)$ to the sphere $\Sigma = \mathbf{r}(D)$ of radius 5, which is considered a two-dimensional subspace of a copy of $\mathbb{R}^3$ with origin at $\mathbf{r}\left(\frac{\pi}{4}, 0.7 \cdot \frac{\pi}{2}\right) \in \Sigma$.

Needless to say, the tangent space is spanned by $\mathbf{r}_u$ and $\mathbf{r}_v$ and is always isomorphic to $\mathbb{R}^2$. Now, it is clear that, at every point, $\mathbf{r}_u \times \mathbf{r}_v$ is orthogonal to the tangent space, which we write either as $\mathbf{r}_u \times \mathbf{r}_v \perp T\Sigma_{\mathbf{x}}$ or as $\mathbf{r}_u \times \mathbf{r}_v \in (T\Sigma_{\mathbf{x}})^{\perp}$ where $(T\Sigma_{\mathbf{x}})^{\perp}$ is the *orthogonal complement* of $T\Sigma_{\mathbf{x}}$, that is, $T\Sigma_{\mathbf{x}} \oplus (T\Sigma_{\mathbf{x}})^{\perp} = \mathbb{R}^3$.

Of course, the geometric vector $\mathbf{r}_u \times \mathbf{r}_v$ at $\mathbf{x} = \mathbf{r}(u,v)$ depends not only upon $\Sigma = \mathbf{r}(D)$, but also on the specific parameterisation $\mathbf{r}$. However, the *standard unit normal*, defined as

$$\widehat{\mathbf{N}} \coloneqq \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

is almost independent upon the parameterisation; indeed, if $\widehat{\mathbf{N}}$ and $\widehat{\mathbf{N}}'$are the standard unit normal of two different f-surfaces $\mathbf{r}$ and $\mathbf{r}'$ with the same image $\Sigma = \mathbf{r}(D) = \mathbf{r}'(D)$ then $\widehat{\mathbf{N}}(u,v) = \pm\widehat{\mathbf{N}}'(u',v')$ at every $\mathbf{x} = \mathbf{r}(u,v) = \mathbf{r}'(u',v')$, which is obvious from the construction of the standard unit normal. (Since $\mathbf{r}_u$ and $\mathbf{r}_v$ are not parallel, the standard unit normal is well-defined.)

The unit normal forms a vector field in $\mathbb{R}^3$ defined on Σ. For instance, the normal vector field to the sphere of radius five is (the usual parameterisation of which is regular everywhere besides at the poles $\theta \in \{0,\pi\}$) is $\widehat{\mathbf{N}}(\theta,\varphi) = (\sin\theta\cos\varphi\,,\sin\theta\sin\varphi\,,\cos\theta)$, and the normal vector field to the circular cylinder of radius five is $(\cos\varphi\,,\sin\varphi\,,0)$, as illustrated below.

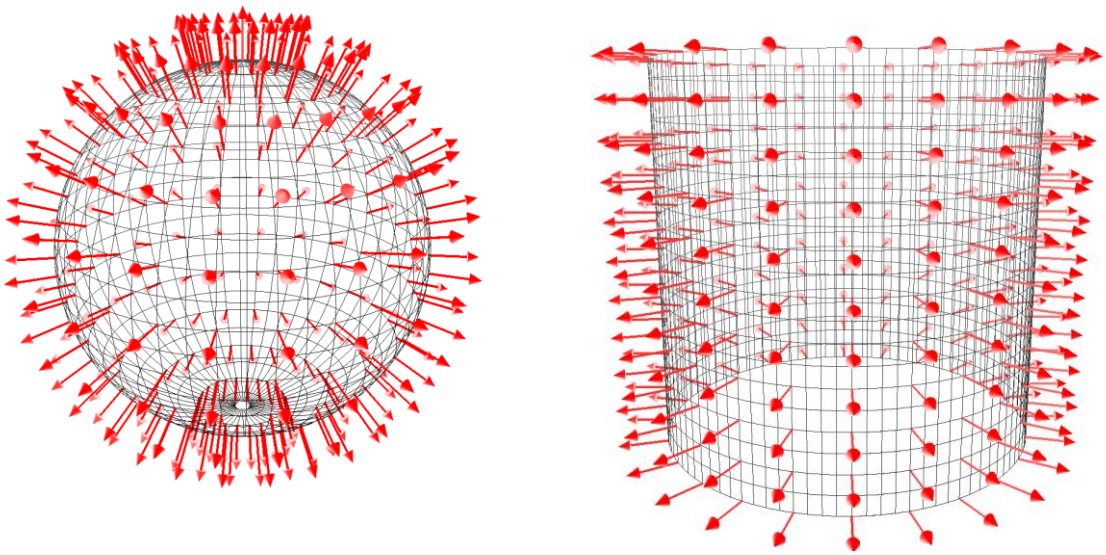


Figure 38. Unit normal fields to a sphere and a circular cylinder.

"Clearly every regular surface has a globally defined smooth unit normal vector field", a novice might say. "Clearly every regular surface has everywhere a *locally* defined unit normal vector field", I would correct him. My proposition is rather immediate. Next we'll see a counter-example to the novice's proposition.

Consider the *Möbius band*, which is the image of

$$\mathbf{r}(u,v) = \underline{\mathbf{e}}\begin{pmatrix} \left(1+\frac{1}{2}v\cos\frac{u}{2}\right)\cos u \\ \left(1+\frac{1}{2}v\cos\frac{u}{2}\right)\sin u \\ \frac{1}{2}v\sin\frac{u}{2} \end{pmatrix}, \qquad (u,v) \in D \coloneqq [0,2\pi[\,\times[-1,1].$$

You can create a Möbius band by taking a strip of paper (30 cm × 2 cm works just fine) and then attach its ends to each other, but, before doing so, you *twist* one of the ends 180°.

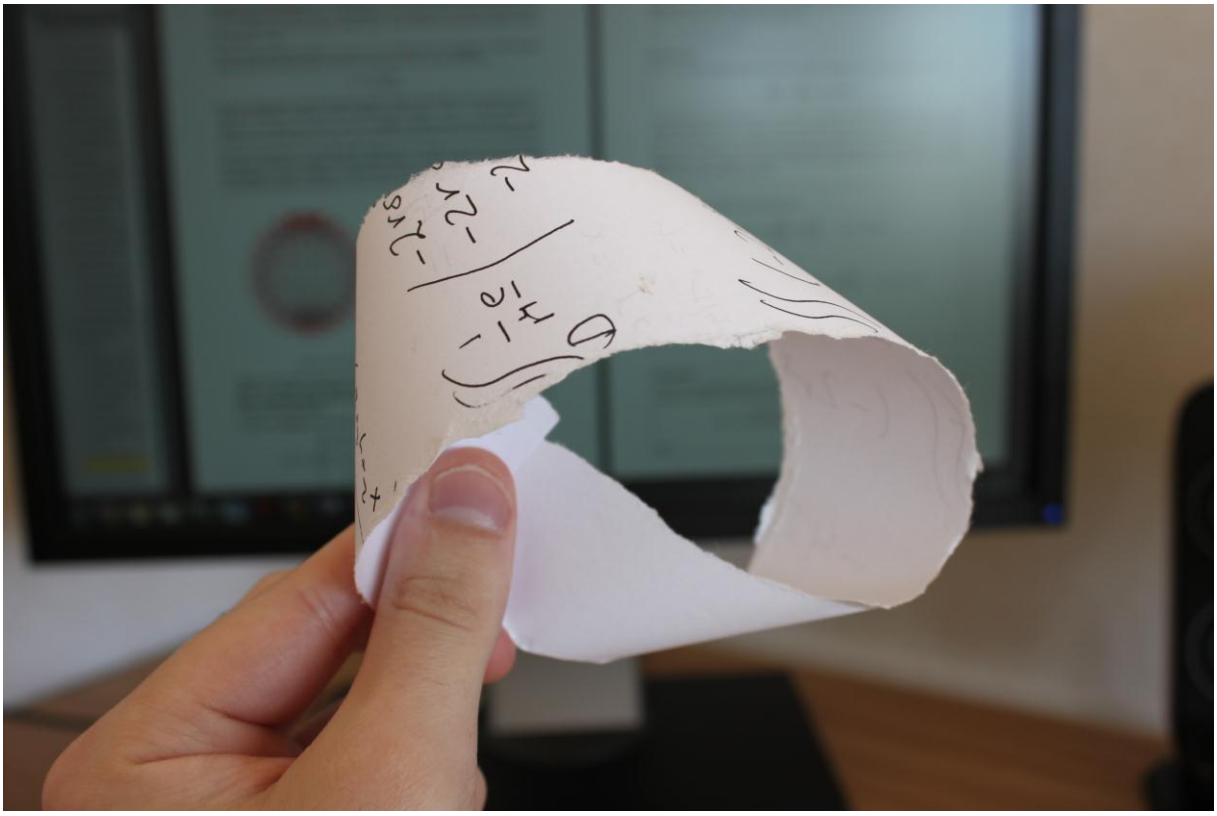


Figure 39. A hand-made Möbius strip.

Below the Möbius strip and its standard normal field $\hat{\mathbf{N}}$ are drawn.

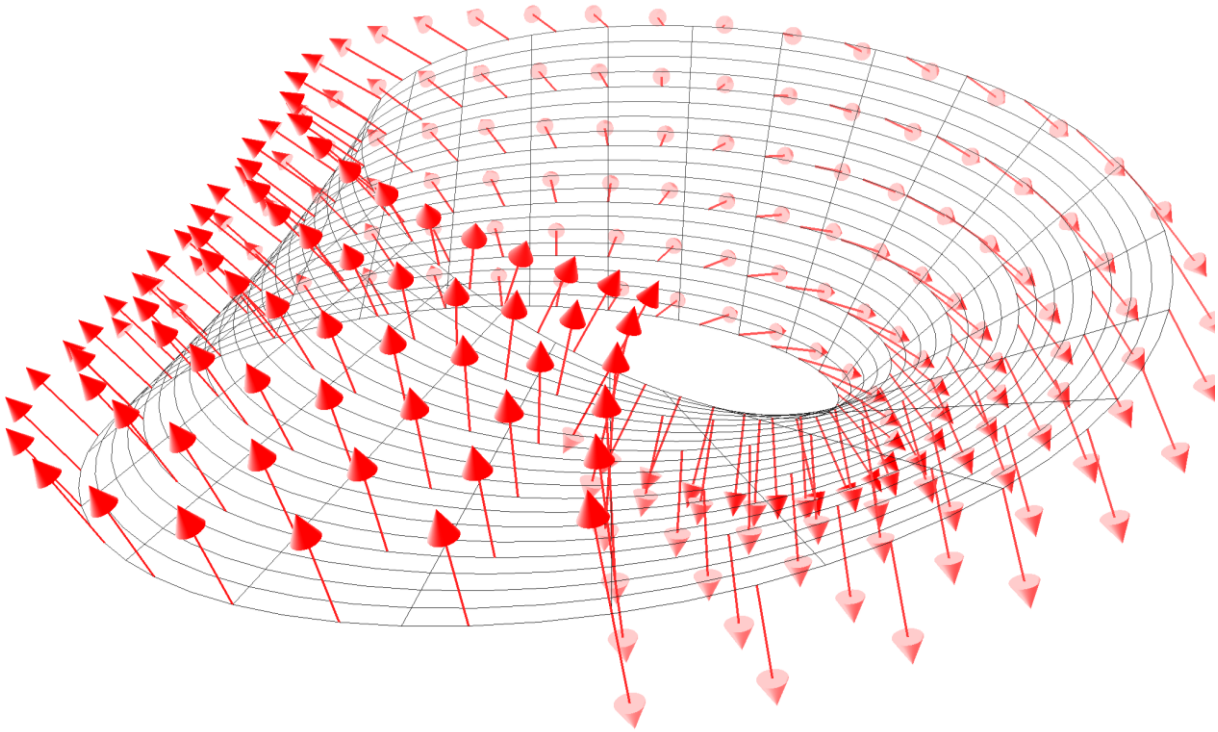


Figure 40. The Möbius strip (or band) is a two-sided surface $\Sigma \subset \mathbb{R}^3$.

4.3.3 The Area of a Surface

Definition NN

Let $\Sigma = \mathbf{r}(D)$ be a regular surface given by a regular f-surface $\mathbf{r}: D \to \mathbb{R}^3$. Then the area of Σ is

$$\iint_\Sigma dA \coloneqq \iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv.$$

The motivation of this definition (as the reader is supposed to know very well) makes clear that the area defined this way coincides with the intuitive notion of the 'area' of a surface, and it is also clear that the area of a regular surface does not depend upon its parameterisation (recall that we do *not* allow non-injective parameterisation functions!). This can be shown by simple application of the 'change of variables' formula for multiple integrals, involving the functionals determinant (or Jacobian), you know.

Example NN

Let $\Pi = \mathbf{r}(D)$ be the (part of a) plane given by

$$\mathbf{r}(u,v) = u\underline{\mathbf{e}}\begin{pmatrix}1\\0\\0\end{pmatrix} + v\underline{\mathbf{e}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad (u,v) \in]0,4[^2.$$

Since

$$\mathbf{r}_u = \underline{\mathbf{e}}\begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \mathbf{r}_v = \underline{\mathbf{e}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad \mathbf{r}_u \times \mathbf{r}_v = \underline{\mathbf{e}}\begin{pmatrix}0\\-1\\1\end{pmatrix}, \qquad |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{2}$$

the area of Π is

$$\iint_\Pi dA \overset{\text{def}}{=} \iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv = \sqrt{2}\iint_D du dv = 16\sqrt{2}.$$

Example NN

We wish to compute the area of one full 'lap' of the helicoid from Example NN. Any such lap is clearly a translation and rotation of any other such lap, such as the surface $\Sigma = \mathbf{r}(D)$ where

$$\mathbf{r}(u,v) = \underline{\mathbf{e}}\begin{pmatrix}2\,u\cos v\\2\,u\sin v\\v\end{pmatrix}, \qquad (u,v) \in D \coloneqq \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\times]0,2\pi[.$$

Now

$$\mathbf{r}_u = \underline{\mathbf{e}}\begin{pmatrix}2\cos v\\2\sin v\\0\end{pmatrix}, \qquad \mathbf{r}_v = \underline{\mathbf{e}}\begin{pmatrix}-2u\sin v\\2u\cos v\\1\end{pmatrix}, \qquad \mathbf{r}_u \times \mathbf{r}_v = \underline{\mathbf{e}}\begin{pmatrix}2\sin v\\-2\cos v\\4u\end{pmatrix},$$
$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{4+16u^2} = 2\sqrt{1+4u^2}.$$

Thus the area

$$\iint_\Sigma dA \overset{\text{def}}{=} \iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv = 2\iint_D \sqrt{1+4u^2}\, du dv = 4\pi \int_{-\pi/2}^{\pi/2} \sqrt{1+4u^2}\, du \approx 76.8.$$

Example NN

Let us compute the area of the Gaussian surface Σ plotted in Example NN.

The parameterisation

$$\mathbf{r}(x,y) = \underline{\mathbf{e}} \begin{pmatrix} x \\ y \\ 5e^{-\frac{1}{8}(x^2+y^2)} \end{pmatrix}, \qquad (x,y) \in D \coloneqq]-10,10[^2$$

implies

$$\mathbf{r}_x = \underline{\mathbf{e}} \begin{pmatrix} 1 \\ 0 \\ -\frac{5x}{4} e^{-\frac{1}{8}(x^2+y^2)} \end{pmatrix}, \qquad \mathbf{r}_y = \underline{\mathbf{e}} \begin{pmatrix} 0 \\ 1 \\ -\frac{5y}{4} e^{-\frac{1}{8}(x^2+y^2)} \end{pmatrix},$$

$$\mathbf{r}_x \times \mathbf{r}_y = \underline{\mathbf{e}} \begin{pmatrix} (5/4)x\, e^{-\frac{1}{8}(x^2+y^2)} \\ (5/4)y\, e^{-\frac{1}{8}(x^2+y^2)} \\ 1 \end{pmatrix},$$

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\frac{25}{16}(x^2+y^2)e^{-\frac{1}{4}(x^2+y^2)} + 1}$$

and so the area

$$\iint_\Sigma dA = \iint_D |\mathbf{r}_x \times \mathbf{r}_y| dx dy \approx 430.6.$$

4.3.4 Curves on Surfaces: The First Fundamental Form

We are interested in measuring the curvature of surfaces. Perhaps the easiest approach to a measure of the curvature of a surface is to study the curvature of space curves that are subsets of the surface. This is what we will do. For simplicity, from now on, we will assume that all f-curves and f-surfaces are regular.

Definition NN

Let $\Sigma = \mathbf{r}(D)$ be a surface given by $\mathbf{r}: D \to \mathbb{R}^3$, and let $\gamma = \mathbf{q}(I)$ be a space curve given by $\mathbf{q}: I \to \mathbb{R}^3$. If $\gamma \subset \Sigma$, we say that γ is a curve *on* Σ and that $\mathbf{q}$ is an f-curve *on* Σ and we may write $\mathbf{q}: I \to \Sigma$.

The following result is immediate.

Observation NN

Let $\Sigma = \mathbf{r}(D)$ be a surface and let $\mathbf{q}: I \to \mathbb{R}^2$ be a plane f-curve with image $\Gamma = \mathbf{q}(I) \subset D$, which we write $\mathbf{q}: I \to D$. Then the space curve $\gamma = \mathbf{p}(I)$ given by $\mathbf{p}: I \to \mathbb{R}^3$ where $\mathbf{p}(t) = \mathbf{r}\big(\mathbf{q}(I)\big)$ is a curve on Σ, that is, $\mathbf{p}: I \to \Sigma$.

In other words, a curve in the parameter region D is mapped, by the f-surface, to a curve on the image of the f-surface. This is how we prefer to specify curves on surfaces. Indeed, to specify a point on the Earth's surface, it is far more natural to specify the longitude and latitude $(u, v) \in D$ in the (very nice, indeed, rectangular) parameter region of the Earth, and not a point $(x, y, z) = \mathbf{r}(D)$ in the subset of the space in which the Earth is embedded.

Every f-surface $\mathbf{r}: D \to \mathbb{R}^3$ has a natural family of curves on it, the so-called *parameter curves.* A parameter curve is the image of a straight line in the parameter region D that is parallel with one of the coordinate axes in $\mathbb{R}^2 \supseteq D$. That is, if (u, v) are coordinates in $\mathbb{R}^2 \supseteq D$, then a general parameter curve can be written as $\mathbf{q}(t) = (u, t)$ for u fixed or $\mathbf{q}(t) = (t, v)$ for v fixed. A common way of drawing a surface $\Sigma = \mathbf{r}(D)$ is to let $E \subset D$ be a *grid* in D and plot $\mathbf{r}(E)$, which then is the union of parameter curves of $\mathbf{r}$. Indeed, this is how all surfaces have been drawn in the examples above. Below a parameter plane grid is mapped to a cylinder via $\mathbf{r}(u, v) = 5(\cos u\,, \sin u\,, v)$.

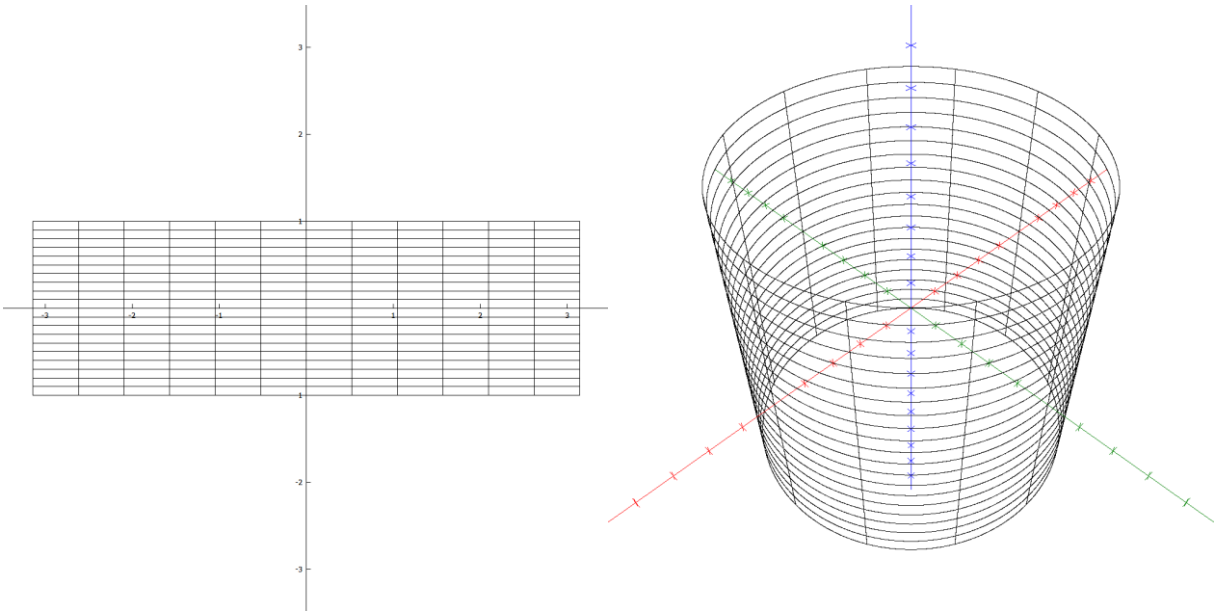


Figure 41. Parameter curves of a cylinder.

Let us now turn to general curves on surfaces. As a simple example, consider the cylinder $\Sigma = \mathbf{r}(D)$ given by

$$\mathbf{r}(\varphi, z) = 5\underline{\mathbf{e}}\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ z\end{pmatrix}, \qquad (\varphi, z) \in D \coloneqq [-\pi, \pi] \times [-1.1, 1.8]$$

and the butterfly curve γ from Example NN but scaled by a factor 0.4 so that $\gamma \subset D$. Then $\mathbf{r}(\gamma) \subset \Sigma$ is a curve on the cylinder, as illustrated below.

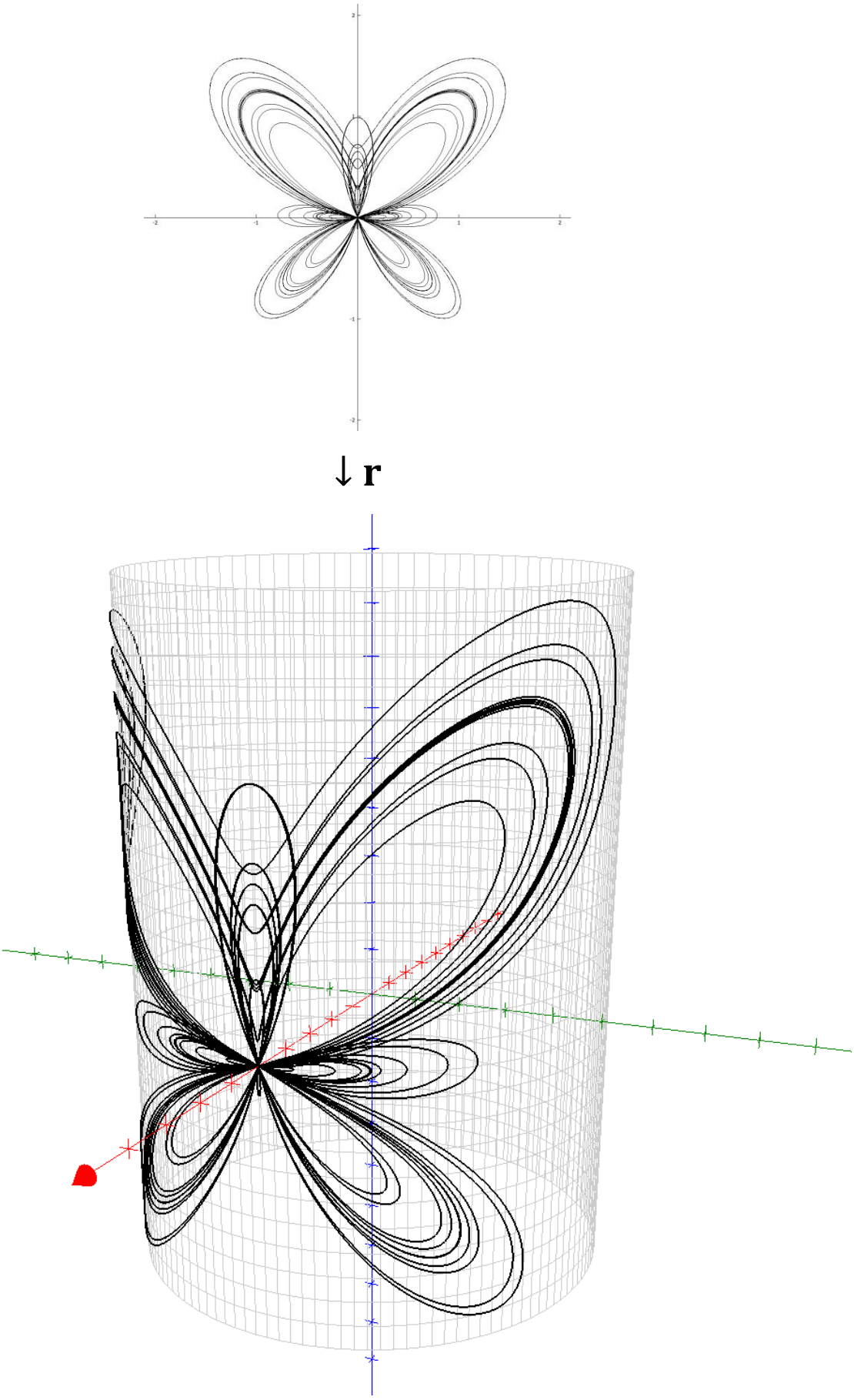


Figure 42. The butterfly curve on a cylinder.

4.3.4.1 Curve Lengths

Let's say that we have a surface $\Sigma = \mathbf{r}(D)$ and a curve $\gamma = \mathbf{r}\big(\mathbf{q}(I)\big)$ on the surface, where $\mathbf{q}\colon I \to D$ is an f-curve on the parameter region D. We wish to find the length L of γ (or, strictly speaking, the length of the f-curve $\mathbf{r} \circ \mathbf{q}\colon I \to \Sigma$ — they coincide unless the f-curve traverses some part of the curve many times).

Let the f-surface be

$$\mathbf{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

and let the f-curve $\mathbf{q}$ be

$$\mathbf{q}(t) = \big(u(t), v(t)\big).$$

Then, by definition,

$$\begin{aligned}
L = \int_I \left|\frac{d}{dt}\mathbf{r}\big(\mathbf{q}(t)\big)\right| dt = \int_I \big|\mathbf{r}'\big(\mathbf{q}(t)\big)\cdot\mathbf{q}'(t)\big| dt = \int_I \left|\underline{\mathbf{e}}\begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}\begin{pmatrix} u_t \\ v_t \end{pmatrix}\right| dt = \\
= \int_I \left|\underline{\mathbf{e}}\begin{pmatrix} x_u u_t + x_v v_t \\ y_u u_t + y_v v_t \\ z_u u_t + z_v v_t \end{pmatrix}\right| dt = \\
= \int_I \sqrt{(x_u u_t + x_v v_t)^2 + (y_u u_t + y_v v_t)^2 + (z_u u_t + z_v v_t)^2}\, dt = \\
= \int_I \sqrt{(x_u^2 + y_u^2 + z_u^2)u_t^2 + 2(x_u x_v + y_u y_v + z_u z_v)u_t v_t + (x_v^2 + y_v^2 + z_v^2)v_t^2}\, dt = \\
= \int_I \sqrt{|\mathbf{r}_u|^2 u_t^2 + 2(\mathbf{r}_u \cdot \mathbf{r}_v)u_t v_t + |\mathbf{r}_v|^2 v_t^2}\, dt = \int_I \sqrt{E u_t^2 + 2F u_t v_t + G v_t^2}\, dt = \\
= \int_I \sqrt{(\dot{\mathbf{q}})^T \mathcal{F} \dot{\mathbf{q}}}\, dt
\end{aligned}$$

where

$$\mathcal{F} \coloneqq \begin{pmatrix} E & F \\ F & G \end{pmatrix} \coloneqq \begin{pmatrix} |\mathbf{r}_u|^2 & \mathbf{r}_u \cdot \mathbf{r}_v \\ \mathbf{r}_u \cdot \mathbf{r}_v & |\mathbf{r}_v|^2 \end{pmatrix}$$

is called *the first fundamental form* of the f-surface $\mathbf{r}$ and $\dot{\mathbf{q}} \coloneqq \frac{d}{dt}\mathbf{q}$.[46]. Notice that we may write, purely formally of course,

[46] If $\mathbf{v} = (v_1, v_2, \dots, v_n) = \underline{\mathbf{e}}\begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}$ is a vector, then $\begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_3 \end{pmatrix}$ is the coordinate matrix of $\mathbf{v}$ relative to the basis $\underline{\mathbf{e}}$. Hence, if we wish to apply a linear transformation A to a vector, as to obtain a new vector, we should write $\underline{\mathbf{e}}AX$ where we explicitly introduce X as the coordinate matrix of $\mathbf{v} = \underline{\mathbf{e}}X$. However, to simplify notation, we will use the same boldface letter to denote both the vector and its coordinate matrix. We let the situation make clear what we mean. For instance, we write simply $A\mathbf{v}$ (since the product of a matrix and a vector is not defined, it is clear that $\mathbf{v}$ denotes the coordinate column matrix). Also, with $\mathbf{v}^T$ we mean the transpose of the coordinate matrix of $\mathbf{v}$, that is, $(v_1 \quad v_2 \quad \cdots \quad v_n)$ (indeed, the transpose of a vector is not defined). In particular, if $\mathbf{v} = \underline{\mathbf{e}}X$ and A is a linear transformation, the we write $\mathbf{v}^T A\mathbf{v}$ instead of $X^T AX$.

$$L = \int_I \sqrt{Eu_t^2 + 2Fu_tv_t + Gv_t^2}dt = \int_I \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\,dt =$$
$$= \int_I \sqrt{E\,du^2 + 2F\,dudv + G\,dv^2} = \int_I ds$$

where ds, of course, represents an infinitesimal arc-length (since, if you integrate it over the curve, you get the total length). Thus, we may write, formally,

$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2 = \begin{pmatrix} du & dv \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}.$$

We summarise:

Definition NN

Let $\mathbf{r}\colon D \to \mathbb{R}^3$ be an f-surface. Then the three functions

$$E \coloneqq |\mathbf{r}_u|^2,$$
$$F \coloneqq \mathbf{r}_u \cdot \mathbf{r}_v,$$
$$G \coloneqq |\mathbf{r}_v|^2$$

(which are scalar fields on D) are called the *coefficients of the first fundamental form* of $\mathbf{r}$, and the matrix

$$\mathcal{F} \coloneqq \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

and also the formal expression

$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2$$

is called *the first fundamental form* of $\mathbf{r}$.

Proposition NN

Let $\Sigma = \mathbf{r}(D)$ be a surface and $\gamma = \mathbf{r}\big(\mathbf{q}(I)\big)$ be a curve on Σ. Then the length of the (f-) curve is

$$L = \int_I \sqrt{(\dot{\mathbf{q}})^T\mathcal{F}\dot{\mathbf{q}}}dt.$$

4.3.4.2 Forms of some Common Surfaces

We will determine the first fundamental forms for a few important f-surfaces.

Example NN

Consider a general plane

$$\mathbf{r}(u, v) = \mathbf{p}_0 + u\mathbf{u} + v\mathbf{v}.$$

We have

$$\mathcal{F} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} |\mathbf{u}|^2 & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{u} \cdot \mathbf{v} & |\mathbf{v}|^2 \end{pmatrix}.$$

Notice that $\mathcal{F}_1$ is always constant, diagonal if $\mathbf{u} \perp \mathbf{v}$, and equal to the identity matrix if also $\mathbf{u}$ and $\mathbf{v}$ are unit vectors. Since *every* plane can be written on the form (↑) with orthogonal and unit $\mathbf{u}$ and $\mathbf{v}$, it follows that *every* plane can be parameterised by an f-plane with $\mathcal{F}_1 = \text{diag}(1, 1)$.

Example NN

Consider the cylinder

$$\mathbf{r}(\varphi, z) = \underline{\mathbf{e}} \begin{pmatrix} a\cos\varphi \\ a\sin\varphi \\ z \end{pmatrix}.$$

Now

$$\mathcal{F} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that the matrix is always constant and diagonal and equal to the identity matrix if $a = \pm 1$.

Example NN

The cone

$$\mathbf{r}(\varphi, z) = \underline{\mathbf{e}} \begin{pmatrix} z\cos\varphi \\ z\sin\varphi \\ z \end{pmatrix}$$

has the diagonal, non-constant first fundamental form

$$\mathcal{F} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Example NN

Consider the 2-sphere of radius a

$$\mathbf{r}(\theta, \varphi) = \underline{\mathbf{e}} \begin{pmatrix} a\sin\theta\cos\varphi \\ a\sin\theta\sin\varphi \\ a\cos\theta \end{pmatrix}.$$

Now

$$\mathcal{F} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}.$$

This is non-constant, but always diagonal.

Example NN

The helicoid might not be a very important surface, but it is fascinating. The parameterisation, which closely resembles that of the cone, is

$$\mathbf{r}(u,v) = \underline{\mathbf{e}}\begin{pmatrix} u\cos v \\ u\sin v \\ v \end{pmatrix}$$

and the first fundamental form is diagonal and non-constant:

$$\mathcal{F} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u^2+1 \end{pmatrix}.$$

Example NN

We end this subsection by finding the first fundamental form of the Gaussian surface from Example NN, parameterised by

$$\mathbf{r}(x,y) = \underline{\mathbf{e}}\begin{pmatrix} x \\ y \\ 5e^{-\frac{1}{8}(x^2+y^2)} \end{pmatrix}.$$

The first fundamental form is thus

$$\mathcal{F} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1+\frac{25x^2}{4}e^{-\frac{1}{4}(x^2+y^2)} & \frac{25xy}{16}e^{-\frac{1}{4}(x^2+y^2)} \\ \frac{25xy}{16}e^{-\frac{1}{4}(x^2+y^2)} & 1+\frac{25y^2}{4}e^{-\frac{1}{4}(x^2+y^2)} \end{pmatrix}.$$

Far from the origin, where x^2+y^2 is big, the surface looks very much like the (flat) plane $\mathbf{r}(x,y) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ with diagonal first fundamental form $\mathrm{diag}(1,1)$, the identity matrix. So we would expect the first fundamental form of the Gaussian to tend to the same diagonal matrix, and, indeed, $\mathcal{F}_1 \to \mathrm{diag}(1,1)$ when $x^2+y^2 \to 1$.

4.3.4.3 Isometries

In this subsection, we are interested in functions $f\colon \Sigma_1 \to \Sigma_2$ between two surfaces Σ_1 and Σ_2. Let $\Sigma_1 = \mathbf{r}_1(D_1)$ and $\Sigma_2 = \mathbf{r}_2(D_2)$. Since we prefer to use surface coordinates to specify points on surfaces, we prefer to work with a function $g\colon D_1 \to D_2$ instead. The relation between f and g is pretty obvious:

$$f = \mathbf{r}_2 \circ g \circ r_1^{-1}$$

where $r_1^{-1}\colon \Sigma_1 \to D_1$ is the inverse of $\mathbf{r}_1\colon D_1 \to \Sigma_1$. Since we are mainly interested in 'nice' functions, we will require f and g to be diffeomorphisms.[47]

[47] Notice that we use non-bold italics to denote functions between two parameter regions (e.g. g) and between two surfaces (e.g. f), even though the image is a vector.

Definition NN

A diffeomorphism $f: \Sigma_1 \to \Sigma_2$ is called an *isometry* if <u>every</u> curve $\gamma \subset \Sigma_1$ is mapped to a curve $f(\gamma) \subset \Sigma_2$ of the same length.

If there exists an isometry $f: \Sigma_1 \to \Sigma_2$ between Σ_1 and Σ_2, then Σ_1 and Σ_2 are said to be *isometric*.

Example NN

Let Σ_1 be the plane $z = 0$, parameterised by $\mathbf{r}_1(u,v) = u\hat{\mathbf{x}} + v\hat{\mathbf{y}} = (u,v)$ where $(u,v) \in D_1 = \mathbb{R}^2$, and let Σ_2 be the unit-radius circular cylinder $x^2 + y^2 = 1$, given parametrically by $\mathbf{r}_2 = (\varphi, z) = (\cos\varphi\,, \sin\varphi\,, z)$ where $(\varphi, z) \in D_2 \coloneqq [0, 2\pi[\times \mathbb{R}$.

The first fundamental forms of Σ_1 and Σ_2 are

$$\mathcal{F}_1 \coloneqq \mathrm{diag}(1,1)\,, \qquad \mathcal{F}_2 \coloneqq \mathrm{diag}(1,1),$$

respectively. Introduce the diffeomorphism

$$g: D_1 \to D_2$$
$$g: (u,v) \mapsto (\varphi, z) = (u,v).$$

with the corresponding diffeomorphism $f = \mathbf{r}_2 \circ g \circ r_1^{-1}$:

$$f: \Sigma_1 \to \Sigma_2$$
$$f: (x,y) \mapsto (\cos x\,, \sin x\,, y)$$

Let $\gamma_1 \coloneqq \mathbf{r}_1\big(\mathbf{q}(I)\big) \subset \Sigma_1$ be any curve on Σ_1. This curve is of length

$$L_1 \coloneqq \int_I \sqrt{(\dot{\mathbf{q}})^T \mathcal{F}_1 \dot{\mathbf{q}}}\, dt.$$

Consider now the curve $\gamma_2 \coloneqq f(\gamma_1) \subset \Sigma_2$, the image of γ on Σ_2, given by f. Since $f = \mathbf{r}_2 \circ g \circ r_1^{-1}$ and $\gamma_1 = \mathbf{r}_1\big(\mathbf{q}(I)\big)$ we have, naturally,

$$\gamma_2 = \mathbf{r}_2(g\left(r_1^{-1}\left(r_1\big(\mathbf{q}(I)\big)\right)\right) = \mathbf{r}_2(g\big(\mathbf{q}(I)\big) = \mathbf{r}_2\big(\mathbf{p}(I)\big)$$

where

$$\mathbf{p} \coloneqq g \circ \mathbf{q}$$

is the f-curve $I \to D_2$. Thus, the image $\mathbf{p}(I) = g\big(\mathbf{q}(I)\big)$ is the curve in the parameter region of the *second* surface that corresponds to the curve $\mathbf{q}(I)$ in the parameter region of the *first* surface. The length of γ_2 is therefore

$$L_2 \coloneqq \int_I \sqrt{(\dot{\mathbf{p}})^T \mathcal{F}_2 \dot{\mathbf{p}}}\, dt = \int_I \sqrt{(\dot{\mathbf{q}})^T \mathcal{F}_2 \dot{\mathbf{q}}}\, dt$$

since $\mathbf{p}(t) = \mathbf{q}(t)$ for every $t \in \mathbb{R}$. But since the plane and the cylinder, parameterised the way we do here, have the same first fundamental forms,

$$L_2 = L_1.$$

Therefore, *any* curve γ_1 on the plane Σ_1 has the same length as its image $\gamma_2 = f(\gamma_1)$ on the cylinder when the mapping f is used. Thus f is an isometry, and the plane and the cylinder are isometric.

The result above is actually obvious. Indeed, draw *any* curve, using a red pencil, on a piece of paper, and then fold the paper to the cylinder. Then, no matter what curve you drew, the curve will have the same length on the resulting cylinder. Hence, a plane and a (circular) cylinder are isometric surfaces. (In fact, the result holds even if you use any other colour than red.) Let us call this procedure the 'kindergarten test of isometry'. The kindergarten test can also show that a plane is isometric to an elliptical cylinder, a circular cone, and even an elliptical cone, as the author just tried.

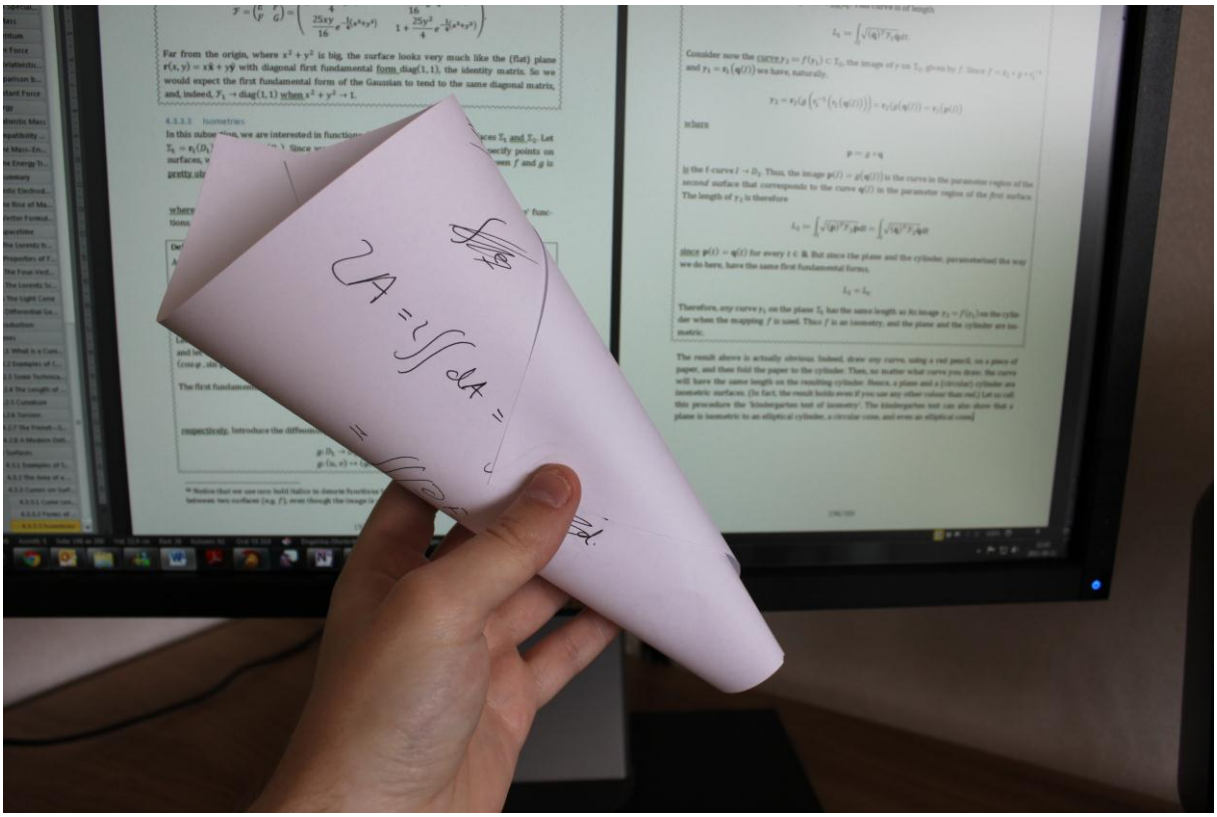


Figure 43. A piece of paper can be folded into a cone, and the length of any curve on the paper will remain unchanged.

The kindergarten test fails for the sphere. Indeed, from experience, we know we cannot deform a (initially flat) piece of paper into a sphere. Thus, we suspect there to be no isometry $\Sigma_1 \to \Sigma_2$ where $\Sigma_1 \subseteq \mathbb{R}^2$ and $\Sigma_2 \subseteq S^2$.

Since curve lengths are computed using the first fundamental form of a surface, the following result should not be that surprising.

Proposition NN

A diffeomorphism $f\colon \Sigma_1 \to \Sigma_2$ is an isometry if and only if the first fundamental forms of $\mathbf{r}_1$ and $f \circ \mathbf{r}_1$ are the same for every f-surface $\mathbf{r}_1$ with image Σ_1.

Proof

⇐) If $\gamma_1 \coloneqq \mathbf{r}_1\big(\mathbf{q}(I)\big)$ is *any* curve on Σ_1, where $\mathbf{r}_1 \colon D \to \mathbb{R}^n$ is any f-surface with image Σ_1, then its length is $L_1 \coloneqq \int_I \sqrt{(\dot{\mathbf{q}})^T \mathcal{F}_1 \dot{\mathbf{q}}}\, dt$ where $\mathcal{F}_1$ is the first fundamental form of $\mathbf{r}_1$. The length of the image $\gamma_2 \coloneqq f(\gamma_1) = f\left(\mathbf{r}_1\big(\mathbf{q}(I)\big)\right) = \mathbf{r}_2\big(\mathbf{q}(I)\big)$, where $\mathbf{r}_2 \coloneqq f \circ \mathbf{r}_1$ is an f-surface with image Σ_2, is $L_2 \coloneqq \int_I \sqrt{(\dot{\mathbf{q}})^T \mathcal{F}_2 \dot{\mathbf{q}}}\, dt$ where $\mathcal{F}_2$ is the first fundamental form of $\mathbf{r}_2 \overset{\text{def}}{=} f \circ \mathbf{r}_1$. By hypothesis, $\mathcal{F}_1 = \mathcal{F}_2$, and so $L_1 = L_2$.

⇒) Conversely, suppose that f is an isometry. This means that, for every curve $\gamma_1 \in \Sigma_1$ of length L_1, the curve $\gamma_2 \coloneqq f(\gamma_1)$ has length L_2 and $L_1 = L_2$. Thus, if you introduce an f-surface $\mathbf{r}_1 \colon D \to \mathbb{R}^3$ such that $\Sigma_1 = \mathbf{r}_1(D)$ and an f-curve $\mathbf{q} \colon I \to D$ such that $\gamma_1 = \mathbf{r}_1\big(\mathbf{q}(I)\big)$, then $L_1 \overset{\text{def}}{=} \int_I \sqrt{(\dot{\mathbf{q}})^T \mathcal{F}_1 \dot{\mathbf{q}}}\, dt$ where $\mathcal{F}_1$ is the first fundamental form of $\mathbf{r}_1$. But since f is an isometry, $L_2 \overset{\text{def}}{=} \int_I \sqrt{(\dot{\mathbf{q}})^T \mathcal{F}_2 \dot{\mathbf{q}}}\, dt = L_1$ where $\mathcal{F}_2$ is the first fundamental form of $\mathbf{r}_2 \coloneqq f \circ \mathbf{r}_1$. Thus $\int_I \sqrt{(\dot{\mathbf{q}})^T \mathcal{F}_1 \dot{\mathbf{q}}}\, dt = \int_I \sqrt{(\dot{\mathbf{q}})^T \mathcal{F}_2 \dot{\mathbf{q}}}\, dt$ for *every* f-curve $\mathbf{q} \colon I \to D$, and one can show that the only possibility is $\mathcal{F}_1 = \mathcal{F}_2$ by studying a set of suitable curves. Thus, for our choice of $\mathbf{r}_1$, the fundamental forms $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathbf{r}_1$ and $f \circ \mathbf{r}_1$ are the same. But since $\mathbf{r}_1$ was completely arbitrary, the equality of the forms must hold for any choice of $\mathbf{r}_1$. ■

The above result agrees with some of our earlier experiences. For instance, we saw that the plane Σ_1 and the (unit) cylinder Σ_2 are isometric, and an isometry is given by

$$f \colon \Sigma_1 \to \Sigma_2$$
$$f \colon (x, y) \mapsto (\cos x\,, \sin x\,, y),$$

and, indeed, our parameterisation of the plane, $\mathbf{r}_1 = u\hat{\mathbf{x}} + v\hat{\mathbf{y}} = (u, v)$ with $ds_1^2 = du^2 + dv^2$ is, via f, transformed to a parameterisation of the cylinder, $\mathbf{r}_2 = (\cos u\,, \sin u\,, v)$ with $ds_2^2 = du^2 + dv^2$; thus $ds_1^2 = ds_2^2$.

4.3.4.4 Angles and Conformality

Let Σ_1 and Σ_2 be surfaces, and let $\gamma_1 \subset \Sigma_1$ and $\gamma_1' \subset \Sigma_1$ be two curves on Σ_1. Let $f \colon \Sigma_1 \to \Sigma_2$ be a diffeomorphism. Then $\gamma_2 \coloneqq f(\gamma_1) \subset \Sigma_2$ and $\gamma_2' \coloneqq f(\gamma_1') \subset \Sigma_2$ are two curves on Σ_2. If $\mathbf{x} \in \gamma_1 \cap \gamma_1'$ is a point of intersection of γ_1 and γ_1', then $f(\mathbf{x}) \in \gamma_2 \cap \gamma_2'$ is a point of intersection of γ_2 and γ_2' [because $\mathbf{x} \in \gamma_1 \cap \gamma_1' \Rightarrow (\mathbf{x} \in \gamma_1) \wedge (\mathbf{x} \in \gamma_1')$ and $\mathbf{x} \in \gamma_1 \Rightarrow f(\mathbf{x}) \in f(\gamma_1)$ and, similarly, $\mathbf{x} \in \gamma_1' \Rightarrow f(\mathbf{x}) \in f(\gamma_1')$. Thus $f(\mathbf{x}) \in f(\gamma_1) \cap f(\gamma_1')$].

At $\mathbf{x} \in \Sigma_1$, the angle α_1 of intersection between γ_1 and γ_1' is defined as the angle between their tangent vectors at $\mathbf{x}$. The angle of intersection α_2 between γ_2 and γ_2' at $f(\mathbf{x}) \in \Sigma_2$ is defined analogously. If f is such that $\alpha_1 = \alpha_2$, that is, if it preserves the angle of intersection at any point of intersection of any pair of curves, then it is said to be *conformal*.

Definition NN

Let $f \colon \Sigma_1 \to \Sigma_2$ be a diffeomorphism. If f preserves angles, then f is *conformal*.

Proposition NN

Let $\Sigma = \mathbf{r}(D)$ be a surface, and let $\gamma = \mathbf{r}\big(\mathbf{q}(I)\big)$ and $\gamma' = \mathbf{r}\big(\mathbf{q}'(I')\big)$ be two curves on Σ. If $\mathbf{x} = \mathbf{r}(u, v) = \mathbf{r}\big(\mathbf{q}(t)\big) = \mathbf{r}\big(\mathbf{q}'(t')\big) \in \gamma \cap \gamma'$ is a point of intersection, then the corresponding angle of intersection is

$$\alpha = \arccos \frac{E\dot{u}\dot{u}' + F(\dot{u}\dot{v}' + \dot{v}\dot{u}') + G\dot{v}\dot{v}'}{\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\sqrt{E\dot{u}'^2 + 2F\dot{u}'\dot{v}' + G\dot{v}'^2}}$$

where

$$\mathbf{q}(t) = \big(u(t), v(t)\big), \qquad \mathbf{q}'(t) = \big(u'(t), v'(t)\big)$$

and

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

is the first fundamental form of $\mathbf{r}$.

Proof

The tangent vector to γ at $\mathbf{x}$ is

$$\mathbf{v}_1 \coloneqq \frac{d}{dt}\Big(\mathbf{r}\big(\mathbf{q}(t)\big)\Big) = \dot{\mathbf{r}}\big(\mathbf{q}(t)\big) \cdot \dot{\mathbf{q}}(t) = \underline{\mathbf{e}}\begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \underline{\mathbf{e}}\begin{pmatrix} x_u\dot{u} + x_v\dot{v} \\ y_u\dot{u} + y_v\dot{v} \\ z_u\dot{u} + z_v\dot{v} \end{pmatrix}$$

where x_u, x_v, … are evaluated at (u, v) and $\dot{u}$ and $\dot{v}$ are evaluated at t. Similarly, the tangent vector to γ' at $\mathbf{x}$ is

$$\mathbf{v}_2 = \underline{\mathbf{e}}\begin{pmatrix} x_u\dot{u}' + x_v\dot{v}' \\ y_u\dot{u}' + y_v\dot{v}' \\ z_u\dot{u}' + z_v\dot{v}' \end{pmatrix}$$

where x_u, x_v, … are evaluated at (u, v) and $\dot{u}'$ and $\dot{v}'$ are evaluated at t'. Thus,

$$\alpha = \arccos\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1||\mathbf{v}_2|} = \arccos\frac{x_u^2\dot{u}\dot{u}' + x_u x_v\dot{u}\dot{v}' + x_u x_v\dot{v}\dot{u}' + x_v^2\dot{v}\dot{v}' + \cdots}{\sqrt{x_u^2\dot{u}^2 + 2x_u x_v\dot{u}\dot{v} + x_v^2\dot{v}^2 + \cdots}\sqrt{x_u^2\dot{u}'^2 + 2x_u x_v\dot{u}'\dot{v}' + x_v^2\dot{v}'^2 + \cdots}} =$$

$$= \arccos\frac{E\dot{u}\dot{u}' + F(\dot{u}\dot{v}' + \dot{v}\dot{u}') + G\dot{v}\dot{v}'}{\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\sqrt{E\dot{u}'^2 + 2F\dot{u}'\dot{v}' + G\dot{v}'^2}}.$$

■

Proposition NN

A diffeomorphism $f\colon \Sigma_1 \to \Sigma_2$ is conformal if and only if the first fundamental forms of $\mathbf{r}_1$ and $f \circ \mathbf{r}_1$ are proportional for every f-surface $\mathbf{r}_1$ with image Σ_1.

Note. Two first fundamental forms $ds_1^2 = E_1du^2 + 2F_1dudv + G_1dv^2$ and $ds_2^2 = E_2du^2 + 2F_2dudv + G_2dv^2$ are proportional iff there exists a scalar field $\Phi\colon D \to \mathbb{R}$ such that $ds_1^2 = \Phi(u, v)ds_2^2$. This implies, in particular, that $E_2 = \Phi(u, v)E_1$. But since E_1 and E_2 both are positive by definition and regularity, it follows that, in fact, $\Phi\colon D \to \mathbb{R}^+$.

Proof

⇐) Let $\mathbf{r}_1$ be an f-surface with image Σ_1, and consider two curves $\gamma_1 = \boldsymbol{r}_1\big(\boldsymbol{q}(I)\big)$ and $\gamma_1' = \mathbf{r}_1\big(\mathbf{q}'(I')\big) \subset \Gamma_1$. Let $\mathbf{x}_1 = \mathbf{r}_1(u, v) = \mathbf{r}_1\big(\mathbf{q}(t)\big) = \mathbf{r}_1\big(\mathbf{q}'(t')\big) \in \gamma_1 \cap \gamma_1'$. The corresponding angle of intersection is

$$\alpha_1 \coloneqq \arccos \frac{E\dot{u}\dot{u}' + F(\dot{u}\dot{v}' + \dot{v}\dot{u}') + G\dot{v}\dot{v}'}{\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\sqrt{E\dot{u}'^2 + 2F\dot{u}'\dot{v}' + G\dot{v}'^2}}.$$

The images $\gamma_2 \coloneqq f(\gamma_1) = f\left(\mathbf{r}_1(\mathbf{q}(I))\right) = \mathbf{r}_2(\mathbf{q}(I))$ and $\gamma_2' \coloneqq f(\gamma_1') = f\left(\mathbf{r}_1(\mathbf{q}'(I'))\right) = \mathbf{r}_2(\mathbf{q}'(I))$ where $\mathbf{r}_2 \coloneqq f \circ \mathbf{r}_1$ is the f-surface with image Σ_2 given by f. Thus, by definition, the angle of intersection of γ_2 and γ_2' at $\mathbf{x}_2 \coloneqq f(\mathbf{x}_1)$ is

$$\alpha_2 \coloneqq \arccos \frac{E_2\dot{u}\dot{u}' + F_2(\dot{u}\dot{v}' + \dot{v}\dot{u}') + G_2\dot{v}\dot{v}'}{\sqrt{E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2}\sqrt{E_2\dot{u}'^2 + 2F_2\dot{u}'\dot{v}' + G_2\dot{v}'^2}}$$

where $ds_2^2 = E_2du^2 + 2F_2dudv + G_2dv^2$ is the second fundamental form of $\mathbf{r}_2$. By assumption, $ds_2^2 = \Phi ds^2$, that is, $E_2 = \Phi E$, $F_2 = \Phi F$, and $G_2 = \Phi G$. Thus,

$$\alpha_2 = \arccos \frac{\Phi E\dot{u}\dot{u}' + \Phi F(\dot{u}\dot{v}' + \dot{v}\dot{u}') + \Phi G\dot{v}\dot{v}'}{\sqrt{\Phi E\dot{u}^2 + 2\Phi F\dot{u}\dot{v} + \Phi G\dot{v}^2}\sqrt{\Phi E\dot{u}'^2 + 2\Phi F\dot{u}'\dot{v}' + \Phi G\dot{v}'^2}} =$$
$$= \frac{\Phi(E\dot{u}\dot{u}' + F(\dot{u}\dot{v}' + \dot{v}\dot{u}') + G\dot{v}\dot{v}')}{\sqrt{\Phi(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)}\sqrt{\Phi\left(E\dot{u}'^2 + 2F\dot{u}'\dot{v}' + G\dot{v}'^2\right)}} = \alpha_1$$

⇒) This is left as an exercise. ■

Corollary NN

Every isometry is conformal.

4.3.4.5 Surface Area

Let Σ be a parameterised surface. Then its area $A = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dudv$ can easily be computed if you know some f-surface $\mathbf{r}\colon D \to \mathbb{R}^3$ with Σ as its image. However, since the first fundamental form is all that is required to measure *distances* on the surface, one would suspect that it also suffices when it comes to area computations. And, indeed, it does.

Proposition NN

Let $\Sigma = \mathbf{r}(D)$. Then $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\det \mathcal{F}} = \sqrt{EG - F^2}$, where $\mathcal{F}$ and $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ is the first fundamental form of $\mathbf{r}$.

Proof

If $\mathbf{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$ then $\mathbf{r}_u = (x_u, y_u, z_u)$, $\mathbf{r}_v = (x_v, y_v, z_v)$ and so, by direct computation,

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\begin{matrix} y_u^2z_v^2 - 2y_uz_vz_uy_v + z_u^2y_v^2 + \\ +z_u^2x_v^2 - 2z_ux_vx_uz_v + x_u^2z_v^2 + = \\ +x_u^2y_v^2 - 2x_uy_vy_ux_v + y_u^2x_v^2 \end{matrix}}$$
$$= \sqrt{(x_u^2 + y_u^2 + z_u^2)(x_v^2 + y_v^2 + z_v^2) - (x_ux_v + y_uy_v + z_uz_v)^2} = \sqrt{EG - F^2}.$$

■

4.3.4.6 Conclusion and Examples

Let $\Sigma = \mathbf{r}(D) \subset \mathbb{R}^3$ be a surface. Traditionally, you would use some parameterisation of Σ, such as $\mathbf{r}$, to compute distances (curve lengths) along the surface, angles, and areas. In this section, we have found an alternative approach to these computations. We have defined three scalar fields E, F, and G on the parameter region D of the surface, and using these fields alone, we can compute distances along curves on Σ, angles between intersecting curves, and surface areas. The descriptions of the curves and subsurfaces are given solely by the (two) surface coordinates in D (and not by the (three) coordinates of the surrounding space), and the results agree with measurements that can be performed using the explicit parameterisation $\mathbf{r}$ of Σ as a subset of $\mathbb{R}^3$. The point with the 'new' approach is that we need not care about the explicit form of $\mathbf{r}$. Indeed, we need not even be 'aware' of the fact that Σ is a subset of some higher-dimensional space, $\mathbb{R}^3$ in this case. All we need to care about is the surface alone, with its coordinates $(u, v) \in D \subset \mathbb{R}^2$.

Our first example shows that our newly-developed machinery actually reduces to well-known formulae in a simple case.

Example NN

A sphere of radius $a > 0$ has a coordinate system with coordinates $(\theta, \varphi) \in D \coloneqq [0, \pi] \times [0, 2\pi[$ and a first fundamental form

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta \, d\varphi^2$$

(that is 'regular' everywhere except at the poles $\theta \in \{0, \pi\}$). Consider the equator γ given by

$$\mathbf{q}(t) = \left(\frac{\pi}{2}, t\right), \qquad t \in [0, 2\pi[.$$

The length of this curve is

$$\int_\gamma ds \coloneqq \int_0^{2\pi} \sqrt{(\dot{\mathbf{q}})^T \mathcal{F} \dot{\mathbf{q}}} dt = \int_0^{2\pi} \sqrt{a^2 \sin^2\theta} \, dt = \int_0^{2\pi} a \sin\frac{\pi}{2} dt = 2\pi a.$$

Consider now the area of the entire surface. This is

$$\iint_D \sqrt{EG - F^2} d\theta d\varphi = \iint_D \sqrt{a^4 \sin^2\theta} \, d\theta d\varphi = \iint_D a^2 \sin\theta \, d\theta d\varphi = a^2 \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi = 4\pi a^2.$$

The reader recognizes the classical volume element

$$dxdy = a^2 \sin\theta \, d\theta d\varphi$$

in spherical coordinates.

Our next examples illustrates the idea that we really only need the first fundamental form to find the metric properties of a surface. We do not even need to be 'aware' of how the surface is embedded in $\mathbb{R}^3$.

Example NN

Consider a surface Σ with coordinates $(u, v) \in D \coloneqq [-10, -10]^2$, and first fundamental form

$$ds^2 = du^2 + e^u dv^2.$$

Consider now the curve $\gamma = \mathbf{q}(I)$ where the f-curve

$$\mathbf{q}(t) = (t, t), \qquad t \in I \coloneqq [-5, 5];$$

thus γ is a straight line in D.

The length of γ, in the coordinate region $D \subset \mathbb{R}^2$, of course, is $10\sqrt{2}$, if we use the usual metric in $\mathbb{R}^2$. However, D is only a mathematical construct, the parameter (or, coordinate) region, and its metric is highly irrelevant. We are interested in the 'physical' length of γ, that is, the length of its 'image' in the 'real' space. Thus, if we happen to know that $\Sigma = \mathbf{r}(D) \subset \mathbb{R}^3$ for some f-surface $\mathbf{r}$, then we are interested in the length of the f-curve $\mathbf{r}\big(\mathbf{q}(I)\big)$ using the metric of $\mathbb{R}^3$.

We don't know of any $\mathbf{r}$, but we do know the first fundamental form of Σ. Thus the sought curve length is

$$\int_\gamma ds \coloneqq \int_{-5}^{5} \sqrt{(\dot{\mathbf{q}})^T \mathcal{F} \dot{\mathbf{q}}}\, dt = \int_{-5}^{5} \sqrt{1 + e^t}\, dt \approx 28.7.$$

If the reader thinks the above example is unnatural and artificial, he might wish to know that it is an immediate preparation for Einstein's general theory of relativity.

4.3.4.7 Some Simple Results

Before we end this subsection, we derive some simple results about curves on surfaces, which we will need later on.

Lemma NN

Let $\gamma = \mathbf{p}(I) = \mathbf{r}\big(\mathbf{q}(I)\big) \subset \Sigma$, $\gamma_1 = \mathbf{p}_1(I_1) = \mathbf{r}\big(\mathbf{q}_1(I_1)\big) \subset \Sigma$, and $\gamma_2 = \mathbf{p}_2(I_2) = \mathbf{r}\big(\mathbf{q}_2(I_2)\big) \subset \Sigma$ be curves on a surface $\Sigma = \mathbf{r}(D)$. Then the following relations hold (pointwise):

(1) $\dot{\mathbf{p}} = \dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v$

(2) $\dot{\mathbf{p}}_1 \cdot \dot{\mathbf{p}}_2 = \dot{\mathbf{q}}_1^T \mathcal{F} \dot{\mathbf{q}}_2$

where $\mathcal{F}$ is the first fundamental form of $\mathbf{r}$.

Proof

(1) By the chain rule, $\dot{\mathbf{p}}(t) = \dot{\mathbf{r}}\big(\mathbf{q}(t)\big) \cdot \dot{\mathbf{q}}(t) = \underline{\mathbf{e}} \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \underline{\mathbf{e}} \begin{pmatrix} x_u \dot{u} + x_v \dot{v} \\ y_u \dot{u} + y_v \dot{v} \\ z_u \dot{u} + z_v \dot{v} \end{pmatrix} = \dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v.$

(2) $\dot{\mathbf{p}}_1 \cdot \dot{\mathbf{p}}_2 = (\dot{u}_1 \mathbf{r}_u + \dot{v}_1 \mathbf{r}_v) \cdot (\dot{u}_2 \mathbf{r}_u + \dot{v}_2 \mathbf{r}_v) = \dot{u}_1 \dot{u}_2 \mathbf{r}_u \cdot \mathbf{r}_u + \dot{u}_1 \dot{v}_2 \mathbf{r}_u \cdot \mathbf{r}_v + \dot{v}_1 \dot{u}_2 \mathbf{r}_v \cdot \mathbf{r}_u + \dot{v}_1 \dot{v}_2 \mathbf{r}_v \cdot \mathbf{r}_v = E\dot{u}_1 \dot{u}_2 + F(\dot{u}_1 \dot{v}_2 + \dot{v}_1 \dot{u}_2) + G\dot{v}_1 \dot{v}_2 = \dot{\mathbf{q}}_1^T \mathcal{F} \dot{\mathbf{q}}_2.$

4.3.5 Surface Curvature – The Second Fundamental Form

How much does a surface, considered a subset of $\mathbb{R}^3$, 'curve', or 'deviate from being a plane'? In this subsection, we will develop a set of measures of surface curvature, at first using a rather intuitive language, but we will end by reformulating our results in a more powerful language that will easily generalise to measures of curvature of general n-dimensional spaces, in a sense that we will describe later. In this section, all curves and surfaces are assumed regular.

Perhaps the most obvious way of measuring the curvature of a surface is to measure the curvature of space curves *on* the surface. Indeed, on a plane $\Pi \subset \mathbb{R}^3$ you can have straight lines $\gamma \subset \Pi$ of zero curvature (and torsion), while this is clearly impossible on a sphere, S^2, say. However, even on a plane Π you *can* have curves of arbitrarily high curvature (but not, of course, torsion). For instance a planar circle of radius r has curvature $1/r$. To resolve this issue, we will separate the curvature of a space curve on a surface into two parts, namely, the *normal curvature* and the *geodesic curvature*.

Let $\gamma = \mathbf{r}\big(\mathbf{q}(I)\big) \subset \Sigma$ be a curve on a surface $\Sigma = \mathbf{r}(D)$. Let $\mathbf{p} \coloneqq \mathbf{r} \circ \mathbf{q}$ be a unit-speed parameterisation function of γ. The unit tangent vector to γ at $\mathbf{x} = \mathbf{r}(u,v) = \mathbf{p}(t)$ is then $\hat{\mathbf{t}}(t) = \dot{\mathbf{p}}(t) \in T\Sigma_{\mathbf{x}}$ in the tangent space of Σ at $\mathbf{x}$. At $\mathbf{x}$, the unit normal vector of Σ is $\widehat{\mathbf{N}}(u,v)$, which is orthogonal to every vector in $T\Sigma_{\mathbf{x}}$; in particular, $\widehat{\mathbf{N}} \perp \dot{\mathbf{p}}(t)$. Therefore the vectors $\widehat{\mathbf{N}}(u,v)$, $\dot{\mathbf{p}}(t)$ and $\widehat{\mathbf{N}}(u,v) \times \dot{\mathbf{p}}(t)$ form a right-handed ON basis at $\mathbf{x}$. The f-curve's acceleration vector at this point is $\ddot{\mathbf{p}}(t) \perp \dot{\mathbf{p}}(t)$. Therefore, $\ddot{\mathbf{p}}(t) \in \operatorname{span}\{\widehat{\mathbf{N}}(u,v), \widehat{\mathbf{N}}(u,v) \times \dot{\mathbf{p}}(t)\}$ and so there are unique scalars $\kappa_n(t)$ and $\kappa_g(t)$ such that

$$\ddot{\mathbf{p}}(t) = \kappa_n(t)\widehat{\mathbf{N}}(u,v) + \kappa_g(t)\widehat{\mathbf{N}}(u,v) \times \dot{\mathbf{p}}(t).$$

These scalars are called the *normal* and *geodesic curvature* of γ at $\mathbf{x}$, respectively. κ_n measures the curvature of γ due to its attachment to the surface Σ, while κ_g, since $\widehat{\mathbf{N}}(u,v) \times \dot{\mathbf{p}}(t) \in T\Sigma_{\mathbf{x}}$, measures the curvature that γ makes inside the surface 'by its own will'. It should be clear that, given a curve $\gamma \subset \Sigma$ and a surface Σ, the normal and geodesic curvatures are well-defined up to their signs, which depend upon the orientations of the surface and the curve.

We summarise this in

Definition NN

Let $\gamma = \mathbf{r}\big(\mathbf{q}(I)\big) = \mathbf{p}(I) \subset \Sigma$ be a curve on a surface $\Sigma = \mathbf{r}(D)$ with $\mathbf{p}$ unit-speed, and write

$$\ddot{\mathbf{p}}(t) = \kappa_n(t)\widehat{\mathbf{N}}(u,v) + \kappa_g(t)\widehat{\mathbf{N}}(u,v) \times \dot{\mathbf{p}}(t).$$

as the f-curve's acceleration vector at $\mathbf{x} = \mathbf{r}(u,v) = \mathbf{p}(t)$. Then the numbers κ_n and κ_g, that are uniquely determined by γ, Σ, and their orientations, are called the *normal* and *geodesic curvatures* of γ at $\mathbf{x}$, respectively.

Pythagoras's theorem yields immediately

Corollary NN

At any point of a curve γ on a surface Γ,

$$\kappa^2 = \kappa_g^2 + \kappa_n^2.$$

In the present subsection, we are mainly interested in the normal curvature, and so we appreciate the simple observation

Corollary NN

Let $\gamma = \mathbf{r}\big(\mathbf{q}(I)\big) = \mathbf{p}(I) \subset \Sigma$ be a curve on a surface $\Sigma = \mathbf{r}(D)$. If $\mathbf{p}$ is unit-speed, then $\kappa_n(t) = \ddot{\mathbf{p}}(t) \cdot \widehat{\mathbf{N}}(u,v)$ at every $\mathbf{x} = \mathbf{r}(u,v) = \mathbf{p}(t)$.

Consider a plane $\Pi \subset \mathbb{R}^3$ and *any* curve $\gamma \subset \Pi$ on it. Both the tangent and normal vectors of the curve are confined to the tangent space of the plane, and so, in particular, the second derivative of any f-curve with image γ will lie in the tangent space. Thus, the normal curvature is identically zero, and the curvature $\kappa = \kappa_g$ is purely geodesic. This motivates why the normal curvatures of curves on a surface are a measure of the surface's curvature. Notice that, at any point $\mathbf{x} \in \Sigma$ on some surface Σ, there are infinitely many curves passing through it, and so there are, in general, infinitely many normal curvature numbers associated with each point $\mathbf{x}$ on a surface. However, if two curves that pass through $\mathbf{x}$ have the same *direction* at $\mathbf{x}$, then they will have the same normal curvature there, as is proved in the following proposition.

Proposition NN

Let $\gamma_1 \subset \Sigma$ and $\gamma_2 \subset \Sigma$ be two curves on a surface Σ, and let $\mathbf{x} \in \gamma_1 \cap \gamma_2$ be a point of intersection of γ_1 and γ_2. If γ_1 and γ_2 have the same instantaneous direction at $\mathbf{x}$, that is, if their unit tangent vectors coincide (or differ by a sign) at $\mathbf{x}$, then their normal curvatures coincide at $\mathbf{x}$, too.

Proof

We will show this by deriving an alternative expression for the normal curvature of a single curve on the surface, and notice that this new formula only depends upon the unit tangent of the curve, from which the proposition follows.

Let $\gamma = \mathbf{r}\big(\mathbf{q}(I)\big) = \mathbf{p}(I) \subset \Sigma$ be a curve on the surface $\Sigma = \mathbf{r}(D)$, with $\mathbf{p}$ unit-speed. At any point $\mathbf{x} = \mathbf{r}(u,v) = \mathbf{p}(t)$ we may write the curve's velocity vector (that is, its unit tangent vector) as

$$\dot{\mathbf{p}}(t) = \dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v$$

according to Lemma NN and so the acceleration vector at this point is

$$\ddot{\mathbf{p}}(t) = \ddot{u}\mathbf{r}_u + \dot{u}(\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}) + \ddot{v}\mathbf{r}_v + \dot{v}(\mathbf{r}_{vu}\dot{u} + \mathbf{r}_{vv}\dot{v}) = \ddot{u}\mathbf{r}_u + \dot{u}^2\mathbf{r}_{uu} + 2\dot{u}\dot{v}\mathbf{r}_{uv} + \ddot{v}\mathbf{r}_v + \dot{v}^2\mathbf{r}_{vv}.$$

Thus, the normal curvature is

$$\kappa_n = \ddot{\mathbf{p}}(t) \cdot \widehat{\mathbf{N}}(u,v) = \dot{u}^2\big(\mathbf{r}_{uu} \cdot \widehat{\mathbf{N}}\big) + 2\dot{u}\dot{v}\big(\mathbf{r}_{uv} \cdot \widehat{\mathbf{N}}\big) + \dot{v}^2\big(\mathbf{r}_{vv} \cdot \widehat{\mathbf{N}}\big)$$

where $\mathbf{r}_u \cdot \widehat{\mathbf{N}} = \mathbf{r}_v \cdot \widehat{\mathbf{N}} = 0$ since $\mathbf{r}_u$ and $\mathbf{r}_v$ are *tangent* to the surface (indeed, $\widehat{\mathbf{N}}$ is defined to be parallel to the cross product of $\mathbf{r}_u$ and $\mathbf{r}_v$). But $\mathbf{r}_{uu}(u,v)$, $\mathbf{r}_{uv}(u,v)$, $\mathbf{r}_{vv}(u,v)$, and $\widehat{\mathbf{N}}(u,v)$ are properties of the f-surface, and so the only piece of information required in order to compute κ_n that comes from the curve is $\big(\dot{u}(t), \dot{v}(t)\big) = \dot{\mathbf{q}}(t)$, the f-curve's tangent vector in the parameter region D of the f-surface. In addition, the tangent vectors $(\dot{u}, \dot{v})$ and $-(\dot{u}, \dot{v}) = (-\dot{u}, -\dot{v})$ yield the same normal curvature κ. ■

Let us now make

Definition NN

Let $\mathbf{r}(D)$ be an f-surface with coordinates $(u,v) \in D$. Then the numbers

$$L \coloneqq \mathbf{r}_{uu} \cdot \widehat{\mathbf{N}}, \qquad M \coloneqq \mathbf{r}_{uv} \cdot \widehat{\mathbf{N}}, \qquad N \coloneqq \mathbf{r}_{vv} \cdot \widehat{\mathbf{N}},$$

which are scalar fields on D, are called the *coefficients of the second fundamental form* of $\mathbf{r}$, and the matrix

$$\mathcal{M} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

is called the *second fundamental form* of $\mathbf{r}$.

Using this terminology, we thus have

Corollary NN

Let $\Sigma = \mathbf{r}(D)$ be a surface with coordinates $(u,v) \in D$, and let $\gamma = \mathbf{r}\big(\mathbf{q}(I)\big) \subset \Sigma$ be the image of a unit-speed f-curve $\mathbf{r} \circ \mathbf{q}$ on Σ. Then the normal curvature of γ at a point $\mathbf{x} = \mathbf{r}(u,v) = \mathbf{r}\big(\mathbf{q}(t)\big)$ is

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 = \dot{\mathbf{q}}^T \mathcal{M} \dot{\mathbf{q}}$$

where $\dot{\mathbf{q}}(t) = \big(\dot{u}(t), \dot{v}(t)\big)$ and L, M, and N are the coefficients of the second fundamental form of $\mathbf{r}$.

4.3.5.1 Forms of some Common Surfaces

Example NN

Consider the plane $\Pi = \mathbf{r}(D)$ where

$$\mathbf{r}(u,v) = \mathbf{p} + u\mathbf{u} + v\mathbf{v}$$

where $\mathbf{p} \in \mathbb{R}^3$ is a fixed point in space, $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^3$ are two orthonormal vectors, and $(u,v) \in D \coloneqq \mathbb{R}^2$. The second fundamental form of $\mathbf{r}$ is

$$\mathcal{M} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, any curve on Σ has necessarily zero normal curvature everywhere.

Example NN

Consider the circular cylinder $\Sigma = \mathbf{r}(D)$ with radius $R > 0$ given by

$$\mathbf{r}(\varphi, z) = \underline{\mathbf{e}} \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}, \qquad (\varphi, z) \in D \coloneqq [0, 2\pi[\times \mathbb{R}.$$

Since (of course!)

$$\widehat{\mathbf{N}} \stackrel{\text{def}}{=} \frac{\mathbf{r}_\varphi \times \mathbf{r}_z}{|\mathbf{r}_\varphi \times \mathbf{r}_z|} = \underline{\mathbf{e}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix},$$

the second fundamental form of $\mathbf{r}$ is

$$\mathcal{M} = \begin{pmatrix} -R & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, a curve $\mathbf{q}: I \to D$ that encircles the cylinder without any vertical motion, that is, a (parameter) curve $\mathbf{q}: t \mapsto \left(\frac{1}{R}t, z\right)$ for some fixed z [the constant $\frac{1}{R}$ is chosen such that $\mathbf{r} \circ \mathbf{q}$ is unit-speed], has the constant normal curvature

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 = L\dot{u}^2 = -R \cdot \frac{1}{R^2} = -\frac{1}{R}.$$

This is obvious, since $\kappa_g = 0$ and so $\kappa_n = \kappa = \frac{1}{r}$ since the curve is a circle of radius r. On the other hand, a curve that 'climbs' the cylinder parallel with its axis, that is, a (parameter) curve $\mathbf{q}: t \mapsto (\varphi, t)$ for some fixed φ [notice that $\mathbf{r} \circ \mathbf{q}$ is unit speed], has always zero normal curvature:

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 = L\dot{u}^2 = -R \cdot 0 = 0.$$

This is obvious, since such a curve is a straight line with $\kappa = 0$, and so, by necessity, $\kappa_n = 0$. (Notice that both curves considered have geodesic curvature $\kappa_g = 0$.)

Example NN, continued

We have seen that at any point $\mathbf{x} = \mathbf{r}(\varphi, z)$ on the cylinder $\Sigma = \mathbf{r}(D)$, a curve on Σ going in the $\mathbf{r}_\varphi$ direction has constant normal curvature $-\frac{1}{R}$, while a curve going in the $\mathbf{r}_z$ direction has zero normal curvature.

What about a curve that goes in some direction in-between? Well, if the (unit-speed) curve's tangent vector is $(\dot{u}, \dot{v})$ in the parameter plane, then its normal curvature is

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 = -R\dot{u}^2 \leq 0.$$

Thus, 0 is the greatest normal curvature that we can possibly obtain. It is also the lowest *absolute* normal curvature that we could possibly obtain. And we do obtain it; this is the curvature of the 'vertical' lines. What is the lowest possible curvature (that is, the *highest* absolute curvature) that we can obtain? Since the absolute curvature increases with increasing $\dot{u}^2$, we wish to find the largest possible $\dot{u}^2$. Since the curve is unit-speed, however, $\dot{u}^2 + \dot{v}^2$ is bounded from above, and so the largest $\dot{u}^2$ is obtained when $\dot{v}^2 = 0$, and the curve is 'horizontal' , with – as we have seen – absolute normal curvature equal to $1/R$.

Thus the directions $\mathbf{r}_\varphi$ and $\mathbf{r}_z$ are highly special, because they correspond to the directions in which a curve has the greatest and lowest absolute normal curvature, respectively. Also, the

normal curvatures $-\frac{1}{R}$ and 0 are special, because they are the normal curvatures with the highest and lowest possible magnitude, respectively.

The directions $\mathbf{r}_\varphi$ and $\mathbf{r}_z$, at every point, are the *principal directions* of the cylinder[48], and the normal curvatures $-1/R$ and 0 are the corresponding *principal curvatures*. We will define these concepts precisely in the next subsection.

4.3.5.2 The Principal Directions and Curvatures

The plane and the cylinder cover both possibilities:

Proposition NN

Let $\Sigma = \mathbf{r}(D)$ be a surface with $\mathbf{r}$ regular. Then, at any $(u, v) \in D$, either

(1) *any* curve $\gamma \in \Sigma$ passing though $\mathbf{r}(u, v)$ will have the same normal curvature at $\mathbf{r}(u, v)$, irrespective of its tangent direction at $\mathbf{r}(u, v)$, or

(2) there exists two distinguished directions[49] [in the tangent plane/space] at $\mathbf{r}(u, v)$, with a right angle between them, such that a curve passing through the point with the first (resp. the other) distinguished direction will have *greater* (resp. *lower*) normal curvature than any other curve passing through the point (that is, in any other direction).

Proof

TBW

Definition NN

Let $\Sigma = \mathbf{r}(D)$ be a surface with $\mathbf{r}$ regular. Then, if $\mathbf{r}(u, v)$ is a point such that every curve that passes through it will have the same normal curvature, the point $\mathbf{r}(u, v)$ is called an *umbilic*, and the normal curvature is called the *principal curvature* at the point. If, on the other hand, there are two distinguished directions in which a curve will have greater and lower (respectively) normal curvature than any other curves through the point, then these two directions are called the *principal directions* and the corresponding normal curvatures are called the *principal curvatures*.

In the case of an umbilic, we often say that *every* direction is a principal direction, because, indeed, a unit-speed curve through the point will have the principal curvature no matter what its direction is. We need a way of computing the principal directions and curvatures of a surface.

Definition NN

Let $\mathcal{F}$ and $\mathcal{M}$ be the fundamental forms of an f-surface. Then the Weingarten matrix is

$$W \coloneqq \mathcal{F}^{-1}\mathcal{M}.$$

[48] It is a 'coincidence' that they, in this case, coincide with the partial derivatives of the parameterisation function $\mathbf{r}$. Their definition relies on the fact that they, at every point, are the directions of greatest and lowest (normal) curvature. Hence, they do not even depend upon the f-surface, only the surface!
[49] We identify directions that are 180° apart.

Proposition NN

Let W be the Weingarten matrix of an f-surface. Then the principal curvatures and directions are the eigenvalues and eigenvectors of W, respectively.

Proof

TBW

Corollary NN

The principal curvatures of an f-surface with fundamental forms $\mathcal{F}$ and $\mathcal{M}$ are the roots of

$$\det(\mathcal{M} - \kappa\mathcal{F}) = 0.$$

If there are two distinct principal curvatures $\kappa_1 \neq \kappa_2$, then the corresponding principal directions are given by (where $X = (\dot{u}, \dot{v})$ is a tangent vector in the parameter plane)

$$(\mathcal{M} - \kappa_i\mathcal{F})X = \mathbf{0}, \qquad i = 1, 2.$$

Proof

TBW

Example NN

Let us verify Proposition NN in the case of the plane and the circular cylinder. In the case of the plane $\Pi = \mathbf{r}(\mathbb{R}^2)$ where $\mathbf{r}(u, v) = \mathbf{x} + u\mathbf{u} + v\mathbf{v}$ with $\mathbf{u}$ and $\mathbf{v}$ ON, the first and second fundamental forms are

$$\mathcal{F} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, the principal curvatures are the roots of

$$\det\begin{pmatrix} -\kappa & 0 \\ 0 & -\kappa \end{pmatrix} = 0,$$

that is, every point is an umbilic with principal curvature $\kappa = 0$.

The cylinder $\Sigma = \mathbf{r}(\mathbb{R}^2)$ of radius $a > 0$where $\mathbf{r}(\varphi, z) = (a\cos\varphi\,, a\sin\varphi\,, z)$ has first and second fundamental forms

$$\mathcal{F} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}.$$

The principal curvatures are therefore given by

$$\det\begin{pmatrix} -a - \kappa a^2 & 0 \\ 0 & -\kappa \end{pmatrix} = 0$$

which is equivalent to

$$\kappa \in \left\{0, -\frac{1}{a}\right\}$$

which we recognizes as the correct principal curvatures of the Σ. The direction corresponding to $\kappa = 0$ is given by

$$\begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{\varphi} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with solution $\dot{\varphi} = 0$. Thus, the corresponding tangent vector on Σ is

$$\underline{\mathbf{e}}\begin{pmatrix} -a\sin\varphi & 0 \\ a\cos\varphi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ k \end{pmatrix} = \underline{\mathbf{e}}\begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$$

for some $k \neq 0$. On the other hand, the direction corresponding to $\kappa = -1/a$ is given by

$$\begin{pmatrix} 0 & 0 \\ 0 & -1/a \end{pmatrix}\begin{pmatrix} \dot{\varphi} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with solution $\dot{z} = 0$. The corresponding tangent vector on Σ is

$$\underline{\mathbf{e}}\begin{pmatrix} -a\sin\varphi & 0 \\ a\cos\varphi & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k \\ 0 \end{pmatrix} = \underline{\mathbf{e}}\begin{pmatrix} -ak\sin\varphi \\ ak\cos\varphi \\ 0 \end{pmatrix}$$

for some $k \neq 0$.

Now we can investigate some more interesting surfaces. Intuitively, one would expect every point on a sphere of radius a to be an umbilic. We will now verify this.

Example NN

The sphere of radius $a > 0$ is the image $\Sigma = \mathbf{r}(D)$ of

$$\mathbf{r}(\theta, \varphi) = \underline{\mathbf{e}}\begin{pmatrix} a\sin\theta\cos\varphi \\ a\sin\theta\sin\varphi \\ a\cos\theta \end{pmatrix}.$$

The first and second fundamental forms are

$$\mathcal{F} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} -a & 0 \\ 0 & -a\sin^2\theta \end{pmatrix}.$$

Thus, the principal curvatures are given by

$$\det\begin{pmatrix} -a - \kappa a^2 & 0 \\ 0 & -a\sin^2\theta - \kappa a^2\sin^2\theta \end{pmatrix} = 0$$

which is equivalent to

$$\kappa = -\frac{1}{a}$$

assuming, as always, that $\theta \notin \{0, pi\}$. Thus every point is an umbilic with the expected normal curvature. (Indeed, there is a unique great *circle* of radius $a > 0$ given a fixed point on the sphere and tangential direction.)

Example NN

Consider the elliptical paraboloid

$$x^2 + y^2 = z$$

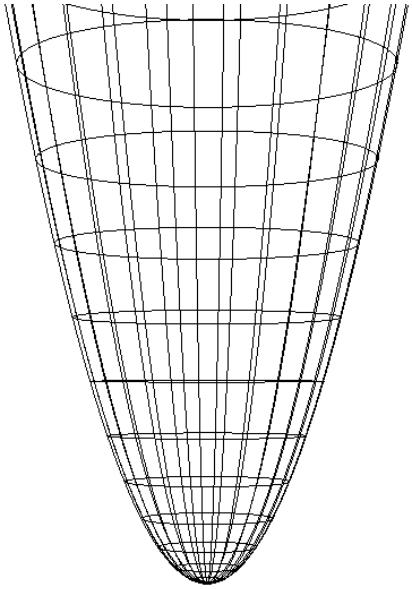
which is the image $\Sigma = \mathbf{r}(D)$ of

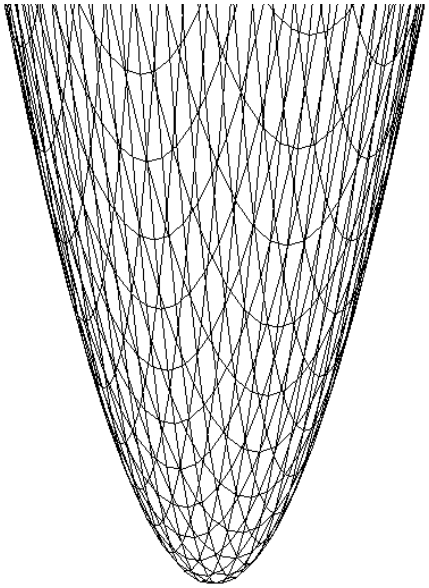
$$\mathbf{r}(u, v) = \underline{\mathbf{e}} \begin{pmatrix} v \cos u \\ v \sin u \\ v^2 \end{pmatrix}, \qquad (u, v) \in D \coloneqq [0, 2\pi[\times [0, \infty[.$$

However, there is a simpler parameterisation for our present needs, namely,

$$\mathbf{r}'(x, y) = \underline{\mathbf{e}} \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix}, \qquad (x, y) \in D' \coloneqq \mathbb{R}^2,$$

and we will use this instead. The (parameter curves of the) lowest part of Σ is shown below using both parameterisations. In any case, the 'vertex' (point of symmetry) is at the origin.





Let us stick to the latter parameterisation. The first and second fundamental forms of $\mathbf{r}$ are then

$$\mathcal{F} = \begin{pmatrix} 1 + 4x^2 & 4xy \\ 4xy & 1 + 4y^2 \end{pmatrix}, \qquad \mathcal{M} = \frac{2}{\sqrt{4x^2 + 4y^2 + 1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider now, in particular, the vertex $(x, y) = (0, 0)$. Here are

$$\mathcal{F}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{M}(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and so the principal curvatures are given by

$$\det \begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = 0$$

which is equivalent to

$$\lambda = 2.$$

Thus, this point is, as we would expect, an umbilic.

We end this section by making

Observation NN

At any point on a surface, the sign of a non-zero principal curvature is $+1$ (resp. -1) iff a curve passing through the point in the direction of the corresponding principal direction curves (that is, has an acceleration vector) in the same (resp. opposite) 'binary direction'[50] as the surface's standard unit normal at that point.

Proof

This is immediate. Indeed, consider a curve $\gamma = \mathbf{p}(I)$ on a surface Σ, that, at some point, is heading in a principal direction. Then the normal curvature κ_n of the curve is simply the corresponding principal curvature, but it is also given by Corollary NN, namely,

$$\kappa_n = \ddot{\mathbf{p}} \cdot \hat{\mathbf{N}}$$

where $\mathbf{p}\colon I \to \Sigma$ is unit-speed. Since, by hypothesis, $\kappa_n \neq 0$, $\ddot{\mathbf{p}}$ and $\hat{\mathbf{N}}$ are not orthogonal (in particular, $\ddot{\mathbf{p}} \neq \mathbf{0}$) and $\kappa_n > 0$ iff $\ddot{\mathbf{p}} \cdot \hat{\mathbf{N}} > 0$ and $\kappa_n < 0$ iff $\ddot{\mathbf{p}} \cdot \hat{\mathbf{N}} < 0$. ■

For example, in the case of the cylinder, a principal curvature at any point is $-1/a < 1$, and, indeed, a curve in the $\mathbf{r}_\varphi$ direction has an acceleration vector pointing *towards* the z-axis, while the standard unit normal is in the direction of $\mathbf{r}_\varphi \times \mathbf{r}_z$, which points *away* from the z-axis. And the sphere has everywhere the principal curvature $-1/a$, and any curve on it will curve towards the origin, while the standard unit normal points away from it.

4.3.5.3 The Gaussian and Mean Curvature

We make

Definition NN

Let κ_1 and κ_2 be the principal curvatures (which are equal at an umbilic) at a point $\mathbf{x} \in \Sigma$. Then

$$K \coloneqq \det W = \kappa_1 \cdot \kappa_2, \text{ and}$$
$$H \coloneqq \operatorname{tr} W = \kappa_1 + \kappa_2$$

are the *Gaussian* and the *mean curvature* of the surface Σ at the point $\mathbf{x}$, respectively.

[50] Two non-orthogonal vectors $\mathbf{a}$ and $\mathbf{b}$ point in the same binary direction if and only if $\mathbf{a} \cdot \mathbf{b} > 0$.

Remark. The mean curvature is also commonly defined as $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ so that it is the true arithmetical mean of the principal curvatures, and not twice this mean. The current author, however, strongly prefers the beauty of the $K = \det W$, $H = \operatorname{tr} W$ formulae.

Before we even motivate the importance of these two concepts (and they are truly important!), let us compute these curvatures for a number of very simple surfaces.

Example NN

Every point in the plane $\mathbf{r}(u, v) = \mathbf{x} + u\mathbf{u} + v\mathbf{v}$ is an umbilic with principal curvature zero. Therefore, both the Gaussian and the mean curvatures are identically zero on the plane.

Example NN

The cylinder $\mathbf{r}(\varphi, z) = (a\cos\varphi\,, a\sin\varphi\,, z)$ has principal curvatures $-1/a$ and 0 at every point. Thus the Gaussian and mean curvatures are 0 and $-1/a$ at every point, respectively.

Example NN

In the sphere (with outward-pointing normal field) of radius a every point is an umbilic with principal curvature $-1/a$. Thus the Gaussian and mean curvatures are identically $1/a^2$ and $-2/a$, respectively.

As usual, it is possible to derive explicit formulae for the Gaussian and mean curvatures. Using the result below, we can compute K and H directly from the fundamental forms; in other words, we don't need to compute the principal curvatures as an intermediate step.

Proposition NN

Let $\mathcal{F}$ and $\mathcal{M}$ be the fundamental forms of an f-surface $\mathbf{r}$. Then the Gaussian and mean curvatures are

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\det \mathcal{M}}{\det \mathcal{F}}, \qquad H = \frac{LG + EN - 2MF}{EG - F^2}.$$

Proof

The principal curvatures are the roots of

$$0 = \det(\mathcal{M} - \kappa\mathcal{F}) = \det\left(\begin{pmatrix} L & M \\ M & N \end{pmatrix} - \kappa \begin{pmatrix} E & F \\ F & G \end{pmatrix}\right) = \det \begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix} =$$
$$= (L - \kappa E)(N - \kappa G) - (M - \kappa F)^2 =$$
$$= (EG - F^2)\kappa^2 + (2MF - LG - EN)\kappa + LN - M^2$$

or, equivalently, since, by regularity, $|\mathbf{r}_u \times \mathbf{r}_v|^2 = EG - F^2 \neq 0$, the roots of

$$\kappa^2 + \frac{2MF - LG - EN}{EG - F^2}\kappa + \frac{LN - M^2}{EG - F^2} = 0.$$

Let the coefficients be $a \coloneqq \frac{2MF - LG - EN}{EG - F^2}$, $b \coloneqq \frac{LN - M^2}{EG - F^2}$. Then

$$\kappa^2 + a\kappa + b = 0.$$

If κ_1 and κ_2 are the principal curvatures (possibly $\kappa_1 = \kappa_2$), then

$$\kappa^2 + a\kappa + b = (\kappa - \kappa_1)(\kappa - \kappa_2)$$

and identification of polynomial coefficients yields

$$H \overset{\text{def}}{=} \kappa_1 + \kappa_2 = -a \overset{\text{def}}{=} \frac{LG + EN - 2MF}{EG - F^2}, \qquad K \overset{\text{def}}{=} \kappa_1\kappa_2 = b \overset{\text{def}}{=} \frac{LN - M^2}{EG - F^2}.$$

■

Given an f-surface $\mathbf{r}$, the tuple fields (κ_1, κ_2) and (K, H) contain exactly the same information. Indeed, Definition NN is a map $(\kappa_1, \kappa_2) \mapsto (K, H)$, and the inverse map is given by, up to the immaterial ordering of the principal curvatures,

Proposition NN

Let K and H be the Gaussian and mean curvatures of an f-surface $\mathbf{r}$, respectively. Then the principal curvatures, at every point, are

$$\kappa_{1,2} = \frac{H}{2} \pm \sqrt{\frac{H^2}{4} - K}.$$

Proof

Trivial, but notice that the relation between the arithmetical mean and the geometric mean yields can be written, assuming the principal curvatures to be non-negative,

$$\frac{\kappa_1 + \kappa_2}{2} \geq \sqrt{\kappa_1\kappa_2}$$

or

$$\frac{H}{2} \geq \sqrt{K}$$

which squares to

$$\frac{H^2}{4} \geq K^2.$$

If exactly one principal curvature is negative, then $K \leq 0$ and so $\frac{H^2}{4} - K$ is clearly non-negative. If both principal curvatures are negative, then $-\kappa_1$ and $-\kappa_2$ are positive, and so

$$\frac{(-\kappa_1) + (-\kappa_2)}{2} \geq \sqrt{\kappa_1\kappa_2}$$

or $-\frac{H}{2} \geq \sqrt{K}$ which also squares to $\frac{H^2}{4} \geq K$. ■

Corollary NN

If K and H are the Gaussian and mean curvatures of some f-surface, then $\frac{H^2}{4} - K \geq 0$.

Example NN

Let us compute the Gaussian curvature for the elliptic paraboloid $x^2 + y^2 = z^2$, given by

$$\mathbf{r}(x,y) = \underline{\mathbf{e}}\begin{pmatrix} x \\ y \\ x^2+y^2 \end{pmatrix}, \qquad (x,y) \in D \coloneqq \mathbb{R}^2.$$

The fundamental forms were found in Example NN, and so Proposition NN yields

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\frac{4}{4x^2+4y^2+1}}{(1+4x^2)(1+4y^2) - 16x^2y^2} = \frac{4}{(4x^2+4y^2+1)^2}.$$

Example NN

Let us now turn to the *hyperbolic* paraboloid $x^2 - y^2 = z$ given by

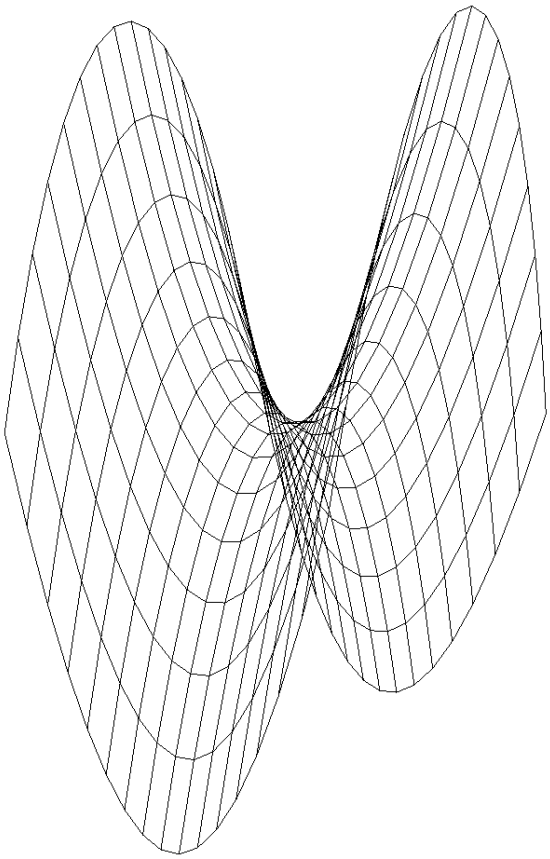
$$\mathbf{r}(x,y) = \underline{\mathbf{e}}\begin{pmatrix} x \\ y \\ x^2-y^2 \end{pmatrix}, \qquad (x,y) \in D \coloneqq \mathbb{R}^2.$$

The fundamental forms are

$$\mathcal{F} = \begin{pmatrix} 1+4x^2 & -4xy \\ -4xy & 1+4y^2 \end{pmatrix}, \qquad \mathcal{M} = \frac{1}{\sqrt{4x^2+4y^2+1}}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

and so the Gaussian curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\frac{-4}{4x^2+4y^2+1}}{(1+4x^2)(1+4y^2) - 16x^2y^2} = \frac{-4}{(4x^2+4y^2+1)^2}.$$



It follows from Observation NN that

Observation NN

At any point on a surface,

(1) $K > 0$ iff $\operatorname{sgn}\kappa_1 = \operatorname{sgn}\kappa_2 \neq 0$, that is, iff *any* curve passing through the point has an acceleration vector with the same 'binary direction' relative to the standard unit normal. There cannot be any curve passing through the point with zero normal curvature.

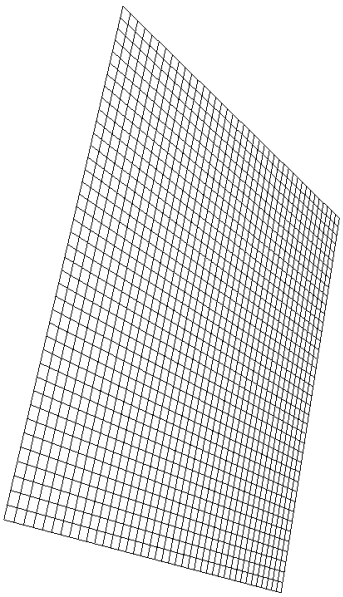
(2) $K < 0$ iff $\operatorname{sgn}\kappa_1 = -\operatorname{sgn}\kappa_2 \neq 0$, that is, iff there are curves passing through the point with the their acceleration vectors in the same 'binary direction' as the standard unit normal, *and*, also, curves that are passing through the point with their acceleration vectors in the opposite 'binary direction'. Assuming only that the normal curvature is a continuous function of the curve's direction at the point, therefore, there are curves passing through the point with zero normal curvature.

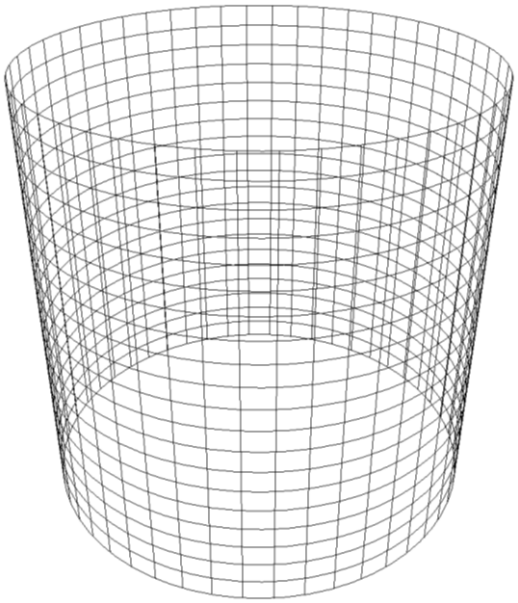
(3) $K = 0$ iff $(\kappa_1 = 0) \vee (\kappa_2 = 0)$. There are curves passing through the point with zero normal curvature. If, in particular, $\kappa_1 = \kappa_2 = 0$, then the surface is a (part of a) plane.

Example NN

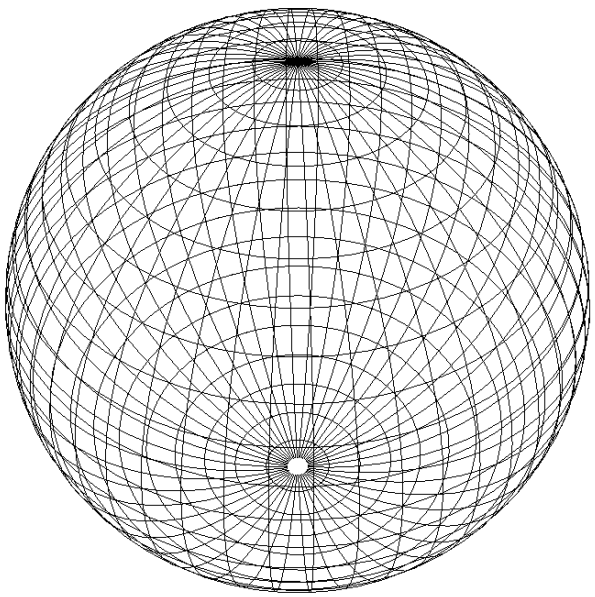
The plane and the cylinder are surfaces with $K = 0$ everywhere, and the sphere and the elliptic paraboloid are surfaces with $K > 0$ everywhere. On the other hand, the hyperbolic paraboloid has $K < 0$ everywhere. Another example of a surface with everywhere negative Gaussian curvature is the hyperboloid $x^2 + y^2 - z^2 = 1$ (exercise). Notice how Observation NN is clearly visible in all these cases:

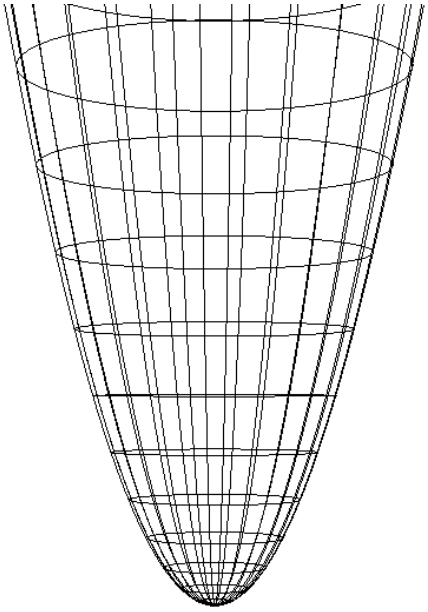
$K = 0$:



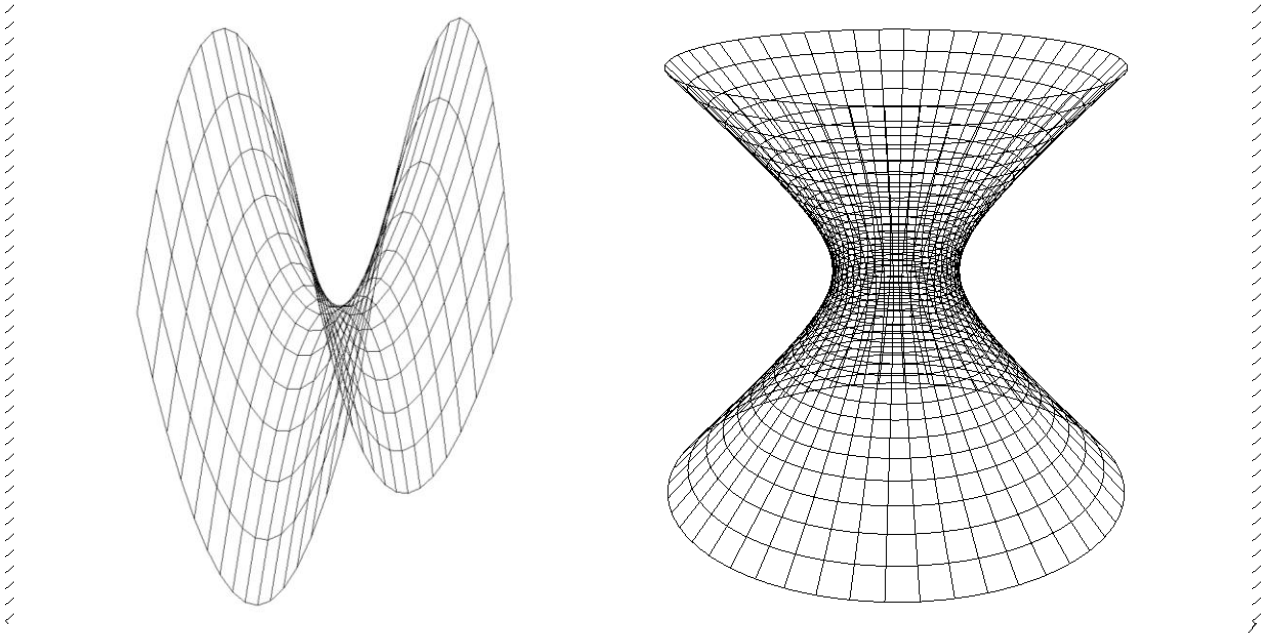


$K > 0$:





$K < 0$:



4.3.5.3.1 Intrinsic and Extrinsic Quantities: The Remarkable Theorem

Consider a bug (ideally 'two-dimensional'), named Buggy, living on a surface Σ. She can measure the distance of any curve on Σ, and the angle of intersection between any two curves. She can also determine the area of any region of Σ. Thus, she can determine the first fundamental form of the surface, using the surface coordinate system of her choice. We therefore say that the first fundamental form is an *intrinsic* property of the surface (and, of course, a surface coordinate system).

On the other hand, Buggy has no idea that Σ is actually a hypersurface of some higher-dimensional space, $\mathbb{R}^3$ in this case. And no matter how she measures distances, angles, and areas (locally), she will not be able to determine if she lives on a plane or an elliptical cylinder, for they are isometric, and so one can always find coordinate systems in both such that their first fundamental forms are identical at corresponding points. Globally, however, a plane and a cylinder are very different. Indeed, if she walks along a straight line from her home, and then eventually comes back to her home, she can pretty much rule out the plane hypothesis. If her house looks the way it looked before, perhaps she is living on a cylinder. On the other hand, if the house is mirrored, perhaps she is living on a Möbius strip. Although these three surfaces are isometric to each other, that is, they share the same local *geometry*, their global properties, or their *topologies*, differ.

The second fundamental form is an *extrinsic* property, since Buggy cannot determine it herself from inside the surface. However, *some* properties of the second fundamental form, she *can* in fact deduce. For instance, it is remarkable that the Gaussian curvature can be expressed in the coefficients of the first fundamental form alone (compare with Proposition NN, which also contains the coefficients of the *second* fundamental form):

Proposition NN

Consider a surface with first fundamental form $\mathcal{F}$ and Gaussian curvature K. Then

$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}.$$

Thus,

Theorem NN (Gauss's *Theorema Egregium*)

The Gaussian curvature of a surface is an *intrinsic* property of the surface.

or, put differently,

The Gaussian curvature is unchanged by an isometry.

The Theorema Egregium has many important applications; one of the most obvious is stated on the Wikipedia page of the theorem[51]:

> "An application of the Theorema Egregium is seen in a common pizza-eating strategy: A slice of pizza can be seen as a surface with constant Gaussian curvature 0. Gently bending a slice must then roughly maintain this curvature (assuming the bend is roughly a local isometry). If one bends a slice horizontally along a radius, non-zero principal curvatures are created along the bend, dictating that the other principal curvature at these points must be zero. This creates rigidity in the direction perpendicular to the fold, an attribute desirable when eating pizza, as it holds its shape long enough to be consumed without a mess."

We are not going to prove the *Theorema Egregium*.

4.3.6 Geodesics

Let $\Sigma = \mathbf{r}(D)$ be a connected surface, and pick any two points $\mathbf{p}_1, \mathbf{p}_2 \in D$ in the parameter plane D. There are infinitely many curves $\gamma = \mathbf{r}(\mathbf{q}(I)) \subset \Sigma$ starting at $\mathbf{x}_1 \coloneqq \mathbf{r}(\mathbf{p}_1)$ and ending at $\mathbf{x}_2 \coloneqq \mathbf{r}(\mathbf{p}_2)$. The length of the shortest such curve is called the *shortest distance* between $\mathbf{x}_1$ and $\mathbf{x}_2$, which, of course, in general, is greater than the Euclidean distance $|\mathbf{x}_1 - \mathbf{x}_2|$ as measured in the ambient $\mathbb{R}^3$. A curve of shortest distance connecting the points is called a *shortest path*. Formally, let (of course, there isn't anything special about the interval $[0, 1]$)

$$\Gamma = \{\mathbf{q}: [0,1] \to D: \mathbf{q}(0) = \mathbf{p}_1, \mathbf{q}(1) = \mathbf{p}_2\}$$

and then

$$d(\mathbf{p}_1, \mathbf{p}_2) = \inf_{\mathbf{q} \in \Gamma} \int_0^1 \sqrt{\dot{\mathbf{q}}^T \mathcal{F} \dot{\mathbf{q}}} dt$$

is the distance between the points $\mathbf{x}_1 \coloneqq \mathbf{r}(\mathbf{p}_1)$ and $\mathbf{x}_2 \coloneqq \mathbf{r}(\mathbf{p}_2)$ with coordinates $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If there is a curve $\mathbf{q} \in \Gamma$ with length $d(\mathbf{p}_1, \mathbf{p}_2)$, then this curve is called a shortest path between the points. On a plane, the shortest distance between two points is along the straight

[51] Theorema Egregium. (2011, January 25). In Wikipedia, The Free Encyclopedia. Retrieved 11:31, September 18, 2011, from http://en.wikipedia.org/w/index.php?title=Theorema_Egregium&oldid=409875634

line connecting the points. This line clearly has zero curvature. In particular, it has zero geodesic curvature everywhere. Any other curve connecting the same pair of points will be longer, and will *not* have zero geodesic curvature everywhere. Consider now a cylinder of radius $a > 0$, with its usual coordinate system, that is, $\mathbf{r}(\varphi, z) = (a\cos\varphi\,, a\sin\varphi\,, z)$. If two points on the cylinder have the same φ or z coordinate, then it is again obvious that the shortest path between them is along a curve of (identically) zero geodesic curvature. Clearly, when it comes to shortest paths, the condition that a curve has identically zero geodesic curvature appears significant. We thus give such a curve a special name:

Definition NN

A curve on a surface is a *geodesic* iff its geodesic curvature is identically zero.

It should be clear that a shortest path between two points is a geodesic, which we will show in a few paragraphs. However, it is easy to see that the converse cannot be true. Consider the following cylinder:

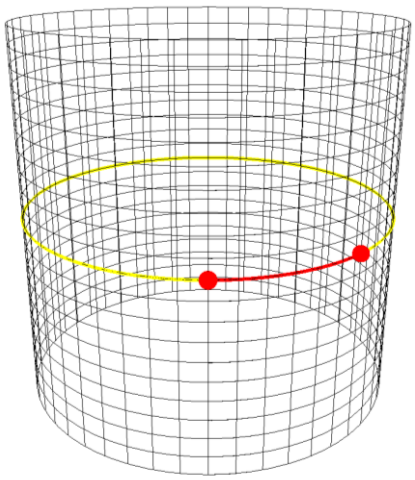


Figure 44. Two geodesics between the same pair of points on a cylinder.

The red curve is a shortest path, and a geodesic. However, the yellow curve is also a geodesic, but obviously not a shortest path. This 'strangeness' is caused by the topology of the cylinder, and one might say that *geodesics are the local shortest paths*. By this we mean, informally, that if we take a geodesic γ between two points and 'deform' it slightly, keeping the end points fixed, then the new curve will *not* be a geodesic, and it will be (at the very least usually!) longer than the original curve (the geodesic).

Also, depending on the topology of a (connected) surface, although any two points does have a shortest distance given by (↑), there might not exist a shortest path, i.e., a curve between the two points with this shortest distance. This possibility is obvious from our careful usage of the infimum and not the minimum. A simple example is the manifold surface $\mathbb{R}^2 \setminus D_r$ where $D_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}, r \in]0, 1[$ and the two points $\mathbf{p}_1 = (-1, 0)$ and $\mathbf{p}_2 = (1, 0)$.

Proposition NN

A curve between two points is a geodesic if and only if the length of a unit-speed f-curve of the curve is stationary (for instance, a local minimum) in the usual sense of the calculus of variations. In addition, the Euler—Lagrange equations are

$$E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2 = \frac{d}{dt}(2E\dot{u} + 2F\dot{v})$$
$$E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2 = \frac{d}{dt}(2F\dot{u} + 2G\dot{v})$$

where $(\dot{u}, \dot{v})$ is the parameter-plane tangent vector of the unit-speed f-curve.

Lemma NN

A unit-speed f-curve $\mathbf{p} = \mathbf{r} \circ \mathbf{q}: I \to \Sigma$ on a surface $\Sigma = \mathbf{r}(D)$ is a geodesic iff $\ddot{\mathbf{p}} \perp \mathbf{r}_v$ and $\ddot{\mathbf{p}} \perp \mathbf{r}_u$ everywhere.

Proof

Since $\mathbf{p}$ is unit-speed, $\ddot{\mathbf{p}} \perp \dot{\mathbf{p}}$, and if $\mathbf{p}$ is also a geodesic, $\ddot{\mathbf{p}} \perp T\Sigma_{\mathbf{p}}$, that is, $\ddot{\mathbf{p}} \perp \mathbf{r}_v$ and $\ddot{\mathbf{p}} \perp \mathbf{r}_u$. Conversely, if $\ddot{\mathbf{p}} \perp \mathbf{r}_v$ and $\ddot{\mathbf{p}} \perp \mathbf{r}_u$ then $\ddot{\mathbf{p}} \perp T\Sigma_{\mathbf{p}}$. Thus, $\mathbf{p}$ is a geodesic, and, in particular, $\ddot{\mathbf{p}} \perp \dot{\mathbf{p}}$, that is, $\mathbf{p}$ is unit-speed. ■

Proof of Proposition NN

Clearly, an f-curve $\mathbf{q}: I \to D$ maximizes/minimizes $\int_0^1 \sqrt{\dot{\mathbf{q}}^T \mathcal{F} \dot{\mathbf{q}}}\, dt$ iff it maximizes/minimizes $\int_0^1 \dot{\mathbf{q}}^T \mathcal{F} \dot{\mathbf{q}}\, dt$. Since the latter expression is simpler, we choose as our Lagrangian the expression

$$L(u, v, \dot{u}, \dot{v}, t) \coloneqq E(u, v)\dot{u}^2 + 2F(u, v)\dot{u}\dot{v} + G(u, v)\dot{v}^2$$

and then the Euler—Lagrange equations

$$\frac{\partial L}{\partial u} = \frac{d}{dt}\frac{\partial L}{\partial \dot{u}}, \qquad \frac{\partial L}{\partial v} = \frac{d}{dt}\frac{\partial L}{\partial \dot{v}}$$

read

$$E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2 = \frac{d}{dt}(2E\dot{u} + 2F\dot{v})$$
$$E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2 = \frac{d}{dt}(2F\dot{u} + 2G\dot{v}).$$

Let $\mathbf{p} \coloneqq \mathbf{r} \circ \mathbf{q}: I \to \Sigma$ be unit-speed, where $\mathbf{q}: D \to \mathbb{R}^3$ is an f-surface for the surface in which the curve lives. Using

$$E \stackrel{\text{def}}{=} \mathbf{r}_u \cdot \mathbf{r}_u, \qquad F \stackrel{\text{def}}{=} \mathbf{r}_u \cdot \mathbf{r}_v, \qquad G \stackrel{\text{def}}{=} \mathbf{r}_v \cdot \mathbf{r}_v$$

we find

$$E_u = 2\mathbf{r}_{uu} \cdot \mathbf{r}_u, \qquad F_u = \mathbf{r}_{uu} \cdot \mathbf{r}_v + \mathbf{r}_u \cdot \mathbf{r}_{vu}, \qquad G_u = 2\mathbf{r}_{uv} \cdot \mathbf{r}_v, \qquad \frac{d}{dt}E = 2\mathbf{r}_u \cdot (\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}),$$
$$\frac{d}{dt}E = 2(\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}) \cdot \mathbf{r}_u, \qquad \frac{d}{dt}F = (\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}) \cdot \mathbf{r}_v + \mathbf{r}_u \cdot (\mathbf{r}_{vu}\dot{u} + \mathbf{r}_{vv}\dot{v})$$

and so the first Euler—Lagrange equation reads, after division by 2,

$$\begin{aligned}(\mathbf{r}_{uu} \cdot \mathbf{r}_u)\dot{u}^2 + (\mathbf{r}_{uu} \cdot \mathbf{r}_v + \mathbf{r}_u \cdot \mathbf{r}_{vu})\dot{u}\dot{v} + (\mathbf{r}_{uv} \cdot \mathbf{r}_v)\dot{v}^2 = \\ = 2(\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}) \cdot \mathbf{r}_u\dot{u} + (\mathbf{r}_u \cdot \mathbf{r}_u)\ddot{u} + (\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}) \cdot \mathbf{r}_v\dot{v} + \mathbf{r}_u \cdot (\mathbf{r}_{vu}\dot{u} + \mathbf{r}_{vv}\dot{v})\dot{v} + \\ + (\mathbf{r}_u \cdot \mathbf{r}_v)\ddot{v}\end{aligned}$$

which is equivalent to

$$0 = \mathbf{r}_{uu}\dot{u}\cdot\mathbf{r}_u\dot{u} + (\mathbf{r}_u\cdot\mathbf{r}_u)\ddot{u} + 2\mathbf{r}_u\cdot\mathbf{r}_{vu}\dot{u}\dot{v} + \mathbf{r}_u\cdot(\mathbf{r}_{vv}\dot{v})\dot{v} + (\mathbf{r}_u\cdot\mathbf{r}_v)\ddot{v}$$

Now, $\mathbf{p}$ is a geodesic iff $\ddot{\mathbf{p}} \perp \mathbf{r}_u$ and $\ddot{\mathbf{p}} \perp \mathbf{r}_v$. The first of these requirements is, using Lemma NN,

$$\frac{d}{dt}(\dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v)\cdot\mathbf{r}_u = 0$$

which expands to

$$\begin{aligned} 0 &= \left(\ddot{u}\mathbf{r}_u + \dot{u}(\mathbf{r}_{uu}\dot{u} + \mathbf{r}_{uv}\dot{v}) + \ddot{v}\mathbf{r}_v + \dot{v}(\mathbf{r}_{vu}\dot{u} + \mathbf{r}_{vv}\dot{v})\right)\cdot\mathbf{r}_u = \\ &= \ddot{u}\mathbf{r}_u\cdot\mathbf{r}_u + \dot{u}^2\mathbf{r}_{uu}\cdot\mathbf{r}_u + \dot{u}\dot{v}\mathbf{r}_{uv}\cdot\mathbf{r}_u + \ddot{v}\mathbf{r}_v\cdot\mathbf{r}_u + \dot{u}\dot{v}\mathbf{r}_{vu}\cdot\mathbf{r}_u + \dot{v}^2\mathbf{r}_{vv}\cdot\mathbf{r}_u. \end{aligned}$$

But this is the exact same equation as (↑), the first Euler—Lagrange equation. Similarly, the second geodesic requirement $\ddot{\mathbf{p}} \perp \mathbf{r}_v$ is seen to be equivalent to the second Euler—Lagrange equation (↑). Thus, a unit-speed f-curve $\mathbf{p}$ on the surface Σ is a geodesic if and only if the Euler—Lagrange equations are satisfied for it. ■

Corollary NN

Given a point $\mathbf{x} \in \Sigma$ and a unit vector $\hat{\mathbf{t}} \in T\Sigma_{\mathbf{x}}$, there is a *unique* unit-speed geodesic passing through $\mathbf{x}$ with unit tangent vector $\hat{\mathbf{t}}$ there.

This follows from the theory of differential equations. We also note that

Corollary NN

If a unit-speed f-curve $\mathbf{r}$ is a shortest path between two points, then $\mathbf{r}$ is a geodesic.

Proof

If $\mathbf{r}$ is a shortest path, then it is a local minimum, and, in particular, a stationary curve with respect to the 'distance' Lagrangian. But then, according to Proposition NN, the f-curve (and its image) is a geodesic. ■

4.3.7 The Covariant Derivative

The concept of a *covariant derivative*, or a '*connection*', is fundamental to modern differential geometry on manifolds. In this subsection, we will introduce a very intuitive and easy-to-grasp (but also restricted) special case of the covariant derivative, namely, the 'metric' (or 'Levi—Civita') connection of a two-dimensional surface in Euclidean $\mathbb{R}^3$.

Consider yourself somewhere on the equator, walking with constant speed along it to the east. You are carrying a flagpole, which you are pointing to the east; the flagpole is representing your velocity vector. You don't understand quite why, but every now and then you walk past a red cottage, and they all look identical to each other. In fact, for some bizarre reason, if you throw an egg at one of the windows of one cottage, then the next cottage will have a splashed egg attached to the corresponding window, and at the very same point of impact! Anyhow, as far as you can tell, you are walking with *constant* velocity; that is, the flagpole is always pointing in the same direction. However, Buggy, who is presently on vacation on the moon, begs to differ. According to her, you are walking in a *circle* with a constant, non-zero, acceleration vector, always pointing towards the centre of the circle. Hence, although you cannot believe it, the flagpole is actually rotating, one full lap every time you walk past a red cottage!

Thus, the derivative of your flagpole (or, more accurately, of your velocity vector) is, of course, non-zero, but to you, it appears to be zero. The *covariant derivative* is a different operator that

will yield the 'derivative' that you experience, being attached to the surface of the Earth. Basically, the covariant derivative is equal to the usual derivative, but also projects the result onto the tangent space of the surface, that is, it removes the component of the derivative that is perpendicular to the surface.

We need the following projection operator.

Definition NN

Let $\Sigma = \mathbf{r}(D) \subset \mathbb{R}^3$ be a surface. Then, for every $\mathbf{x} \in \Sigma$, the operator $\pi_{\mathbf{x}}\colon \mathbb{R}^3 \to T\Sigma_{\mathbf{x}}$ is given by

$$\pi_{\mathbf{x}}(\mathbf{v}) \coloneqq \mathbf{v} - \big(\widehat{\mathbf{N}} \cdot \mathbf{v}\big)\widehat{\mathbf{N}}, \qquad \forall \mathbf{v} \in \mathbb{R}^3$$

where $\widehat{\mathbf{N}}$ is the standard unit normal of Σ at $\mathbf{x}$.

Proposition NN

Fix some $\mathbf{x} \in \Sigma$, and consider the operator $\pi_{\mathbf{x}}$. For all $\mathbf{v}, \mathbf{v}' \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$,

(1) $\mathbf{v} \in T\Sigma_{\mathbf{x}} \Rightarrow \pi_{\mathbf{x}}(\mathbf{v}) = \mathbf{v}$,

(2) $\mathbf{v} \in (T\Sigma_{\mathbf{x}})^{\perp} \Rightarrow \pi_{\mathbf{x}}(\mathbf{v}) = \mathbf{0}$,

(3) $\pi_{\mathbf{x}}(\mathbf{v}) \in T\Sigma_{\mathbf{x}}$,

(4) $\pi_{\mathbf{x}}\big(\pi_{\mathbf{x}}(\mathbf{v})\big) = \pi_{\mathbf{x}}(\mathbf{v})$ (idempotence),

(5) $\begin{cases} \pi_{\mathbf{x}}(\mathbf{v} + \mathbf{v}') = \pi_{\mathbf{x}}(\mathbf{v}) + \pi_{\mathbf{x}}(\mathbf{v}') \\ \pi_{\mathbf{x}}(\alpha\mathbf{v}) = \alpha\pi_{\mathbf{x}}(\mathbf{v}) \end{cases}$ (linearity), and

(6) $\pi_{\mathbf{x}}$ is a projection.

Proof

(1) If $\mathbf{v} \in T\Sigma_{\mathbf{x}}$ then $\mathbf{v} = a\mathbf{r}_u + b\mathbf{r}_v$ for some numbers a and b. Thus $\widehat{\mathbf{N}} \cdot \mathbf{v} = \left(\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right) \cdot (a\mathbf{r}_u + b\mathbf{r}_v) = 0$ and so $\pi_{\mathbf{x}}(\mathbf{v}) = \mathbf{v} - \big(\widehat{\mathbf{N}} \cdot \mathbf{v}\big)\widehat{\mathbf{N}} = \mathbf{v} - \mathbf{0} = \mathbf{v}$.

(2) If $\mathbf{v} \in (T\Sigma_{\mathbf{x}})^{\perp}$ then $\mathbf{v} = k\widehat{\mathbf{N}}$ for some number k. Thus $\pi_{\mathbf{x}}(\mathbf{v}) \coloneqq k\widehat{\mathbf{N}} - \big(\widehat{\mathbf{N}} \cdot k\widehat{\mathbf{N}}\big)\widehat{\mathbf{N}} = k\widehat{\mathbf{N}} - k\widehat{\mathbf{N}} = \mathbf{0}$.

(3) $\pi_{\mathbf{x}}(\mathbf{v}) \cdot \widehat{\mathbf{N}} = \big(\mathbf{v} - \big(\widehat{\mathbf{N}} \cdot \mathbf{v}\big)\widehat{\mathbf{N}}\big) \cdot \widehat{\mathbf{N}} = \mathbf{v} \cdot \widehat{\mathbf{N}} - \big(\widehat{\mathbf{N}} \cdot \mathbf{v}\big)\widehat{\mathbf{N}} \cdot \widehat{\mathbf{N}} = \mathbf{v} \cdot \widehat{\mathbf{N}} - \widehat{\mathbf{N}} \cdot \mathbf{v} = 0$. Thus $\pi_{\mathbf{x}}(\mathbf{v}) \in T\Sigma_{\mathbf{x}}$ because $T\Sigma_{\mathbf{x}} \oplus (T\Sigma_{\mathbf{x}})^{\perp} = \mathbb{R}^3$ and $(T\Sigma_{\mathbf{x}})^{\perp} = \operatorname{span}\{\widehat{\mathbf{N}}\}$.

(4) Since $\pi_{\mathbf{x}}(\mathbf{v}) \in T\Sigma_{\mathbf{x}}$ for all $\mathbf{v} \in \mathbb{R}^3$ and $\pi_{\mathbf{x}}(\mathbf{v}) = \mathbf{v}$ for every $\mathbf{v} \in T\Sigma_{\mathbf{x}}$, it follows that $\pi_{\mathbf{x}}\big(\pi_{\mathbf{x}}(\mathbf{v})\big) = \pi_{\mathbf{x}}(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^3$.

(5) $\pi_{\mathbf{x}}(\mathbf{v} + \mathbf{v}') \coloneqq (\mathbf{v} + \mathbf{v}') - \big(\widehat{\mathbf{N}} \cdot (\mathbf{v} + \mathbf{v}')\big)\widehat{\mathbf{N}} = \mathbf{v} - \big(\widehat{\mathbf{N}} \cdot \mathbf{v}\big)\widehat{\mathbf{N}} + \mathbf{v}' - \big(\widehat{\mathbf{N}} \cdot \mathbf{v}'\big)\widehat{\mathbf{N}} = \pi_{\mathbf{x}}(\mathbf{v}) + \pi_{\mathbf{x}}(\mathbf{v}')$ while $\pi_{\mathbf{x}}(\alpha\mathbf{v}) = \alpha\mathbf{v} - \big(\widehat{\mathbf{N}} \cdot \alpha\mathbf{v}\big)\widehat{\mathbf{N}} = \alpha\big(\mathbf{v} - \big(\widehat{\mathbf{N}} \cdot \mathbf{v}\big)\widehat{\mathbf{N}}\big) = \alpha\pi_{\mathbf{x}}(\mathbf{v})$.

(6) By definition, a projection is an idempotent linear operator. ■

Definition NN

Let $\gamma = \mathbf{p}(I) = \mathbf{r}\big(\mathbf{q}(I)\big)$ be a curve on a surface $\Sigma = \mathbf{r}(D)$, and let $\mathbf{u}\colon I \to \mathbb{R}^3$ be a function that assigns a Σ-*surface-tangential* vector to every point on γ, that is, $\mathbf{u}(t) \in T\Sigma_{\mathbf{p}(t)}$ for every $t \in I$. [For instance, $\mathbf{u}$ might be the velocity vector $\dot{\mathbf{p}}$.] Then $\dot{\mathbf{u}}(t) = \frac{d}{dt}\mathbf{u}(t)$ is the usual rate of change of $\mathbf{u}(t)$ with respect to t, and

$$\frac{\nabla \mathbf{u}}{dt} \coloneqq \pi_{\mathbf{p}(t)}\big(\dot{\mathbf{u}}(t)\big),$$

is called the *covariant derivative* of $\mathbf{u}$ along the f-curve $\mathbf{p}$ at time t .

Although $\mathbf{u}(t) \in T\Sigma_{\mathbf{p}(t)}$ for every $t \in I$, the usual derivative $\dot{\mathbf{u}}(t)$ need not be Σ-surface-tangential. However, the covariant derivative is by Lemma NN(3). We prove the linearity and Leibniz properties of ∇/dt:

Proposition NN

Let $\gamma = \mathbf{p}(I) = \mathbf{r}\big(\mathbf{q}(I)\big)$ be a curve on a surface $\Sigma = \mathbf{r}(D)$ and let $\mathbf{u}$ and $\mathbf{u}'\colon I \to \mathbb{R}^3$ be two functions that assign Σ-surface-trangential vectors to every point on γ. Then, for any constant scalars $\alpha, \beta \in \mathbb{R}$, the function $\alpha\mathbf{u} + \beta\mathbf{u}'\colon I \to \mathbb{R}^3$, defined by $(\alpha\mathbf{u} + \beta\mathbf{u}')(t) \coloneqq \alpha\mathbf{u}(t) + \beta\mathbf{u}'(t)$ for all $t \in I$ is also a function that assigns a Σ-surface-tangential vector to every point on γ, and

$$\frac{\nabla(\alpha\mathbf{u} + \beta\mathbf{u}')}{dt} = \alpha\frac{\nabla\mathbf{u}}{dt} + \beta\frac{\nabla\mathbf{u}'}{dt}.$$

If, in addition, $f\colon I \to \mathbb{R}$ then $f\mathbf{u}\colon I \to \mathbb{R}^3$ defined by $(f\mathbf{u})(t) \coloneqq f(t)\mathbf{u}(t)$ for all $t \in I$ is also a function that assigns a Σ-surface-tangential vector to every point on γ, and

$$\frac{\nabla(f\mathbf{u})}{dt} = \frac{df}{dt}\cdot\mathbf{u} + f\frac{\nabla\mathbf{u}}{dt}.$$

Proof

For any $t \in I$, $\mathbf{u}(t) \in T\Sigma_{\mathbf{p}(t)}$ and $\mathbf{u}'(t) \in T\Sigma_{\mathbf{p}(t)}$. But since $T\Sigma_{\mathbf{p}(t)}$ is a vector space, $(\alpha\mathbf{u} + \beta\mathbf{u}')(t) \overset{\text{def}}{=} \alpha\mathbf{u}(t) + \beta\mathbf{u}'(t) \in T\Sigma_{\mathbf{p}(t)}$, too. Furthermore,

$$\begin{aligned}\frac{\nabla(\alpha\mathbf{u} + \beta\mathbf{u}')}{dt} &= \pi_{\mathbf{p}(t)}\left(\frac{d}{dt}[\alpha\mathbf{u}(t) + \beta\mathbf{u}'(t)]\right) = \pi_{\mathbf{p}(t)}\big(\alpha\dot{\mathbf{u}}(t) + \beta\dot{\mathbf{u}}'(t)\big) = \\ &= \alpha\pi_{\mathbf{p}(t)}\big(\dot{\mathbf{u}}(t)\big) + \beta\pi_{\mathbf{p}(t)}\big(\dot{\mathbf{u}}'(t)\big) = \alpha\frac{\nabla\mathbf{u}}{dt} + \beta\frac{\nabla\mathbf{u}'}{dt}\end{aligned}$$

using Lemma NN(5). If $f\colon I \to \mathbb{R}$ then obviously $f\mathbf{u}\colon I \to \mathbb{R}^3$ and $(f\mathbf{u})(t) \in T\Sigma_{\mathbf{p}(t)}$ for every $t \in I$, and

$$\begin{aligned}\frac{\nabla(f\mathbf{u})}{dt} &= \pi_{\mathbf{p}(t)}\left(\frac{d}{dt}\big(f(t)\mathbf{u}(t)\big)\right) = \pi_{\mathbf{p}(t)}\left(\frac{df}{dt}\mathbf{u} + f\frac{d\mathbf{u}}{dt}\right) = \frac{df}{dt}\pi_{\mathbf{p}(t)}(\mathbf{u}) + f\pi_{\mathbf{p}(t)}\left(\frac{d\mathbf{u}}{dt}\right) = \\ &= \frac{df}{dt}\cdot\mathbf{u} + f\frac{\nabla\mathbf{u}}{dt}\end{aligned}$$

since $\mathbf{u}(t) \in T\Sigma_{\mathbf{p}(t)}$. ∎

Example NN

You are walking eastwards along the equator of the Earth with constant speed. Assume that the Earth is the image $\Sigma = \mathbf{r}(D) \subset \mathbb{R}^3$ of

$$\mathbf{r} = \underline{\mathbf{e}} \begin{pmatrix} R \sin\theta \cos\varphi \\ R \sin\theta \sin\varphi \\ R \cos\theta \end{pmatrix}$$

and your trajectory $\gamma = \mathbf{r}\big(\mathbf{q}(I)\big)$ is given by

$$\mathbf{q}(t) = \left(\frac{\pi}{2}, t\right) \in D;$$

that is, you are at

$$\mathbf{p}(t) \coloneqq \mathbf{r}\big(\mathbf{q}(t)\big) = \underline{\mathbf{e}} \begin{pmatrix} R \cos t \\ R \sin t \\ 0 \end{pmatrix} \in \Sigma \subset \mathbb{R}^3$$

at time t. Considered as a space curve, your velocity vector is clearly

$$\dot{\mathbf{p}}(t) = \underline{\mathbf{e}} \begin{pmatrix} -R \sin t \\ R \cos t \\ 0 \end{pmatrix}$$

which is nonconstant. Indeed, the acceleration

$$\ddot{\mathbf{p}}(t) = \underline{\mathbf{e}} \begin{pmatrix} -R \cos t \\ -R \sin t \\ 0 \end{pmatrix} = -R\hat{\mathbf{N}},$$

where $\hat{\mathbf{N}}$ is the Earth's standard unit normal at the current position, is utterly nonzero and pointing towards the origin. However, the covariant derivative of your velocity vector is

$$\frac{\nabla \dot{\mathbf{p}}}{dt} \overset{\text{def}}{=} \pi_{\mathbf{x}}\big(\ddot{\mathbf{p}}(t)\big) = \ddot{\mathbf{p}}(t) - \Big(\hat{\mathbf{N}} \cdot \ddot{\mathbf{p}}(t)\Big)\hat{\mathbf{N}} = -R\hat{\mathbf{N}} - \Big(\hat{\mathbf{N}} \cdot \big(-R\hat{\mathbf{N}}\big)\Big)\hat{\mathbf{N}} = -R\hat{\mathbf{N}} - (-R)\hat{\mathbf{N}} = \mathbf{0}.$$

Thus, the covariant derivative 'works'!

If it would happen that the *ordinary* derivative $\dot{\mathbf{u}}$ of a vector defined along a curve on the surface is surface-tangential, then the covariant derivative $\nabla\mathbf{u}/dt$ is equal to the ordinary derivative, by Definition NN and Lemma NN(1). Thus, we might say that the covariant derivative removes the features of the ordinary derivative that are not visible to a (two-dimensional) person living on the surface, while retaining all other properties.

4.3.7.1 The Christoffel Symbols

We want to find an alternative expression for the covariant derivative. There are basically two reasons for this. First, the definition (↑) in its explicit form (↑) is not very convenient. Second, this alternate expression is easily generalizable to general manifolds (such as 'surfaces' that are *not* embedded in some higher-dimensional space (such as $\mathbb{R}^3$), in which case there is no 'stand-

ard unit normal' field $\widehat{\mathbf{N}}$ at all), as we will see in the next chapter. We will base our alternative expression on the so-called *Christoffel symbols* of the f-surface.

Let $\mathbf{r}\colon D \to \mathbb{R}^3$ be an f-surface with image $\Sigma = \mathbf{r}(D)$ and coordinates $(u, v) \in D$. Then $\mathbf{r}_u$ and $\mathbf{r}_v$ are vector fields on Σ. Of course, the usual derivatives $\mathbf{r}_{uu}$, $\mathbf{r}_{uv} = \mathbf{r}_{vu}$, and $\mathbf{r}_{vv}$ are not Σ-surface-tangential, in general. We might consider the corresponding covariant derivatives, instead. Since, at any point $\mathbf{x} \in \Sigma$, the vectors $\mathbf{r}_u$, $\mathbf{r}_v$ and $\widehat{\mathbf{N}}$ form a basis, we may write

$$\begin{aligned}\mathbf{r}_{uu} &= \Gamma_{11}^1\mathbf{r}_u + \Gamma_{11}^2\mathbf{r}_v + A\widehat{\mathbf{N}}\\ \mathbf{r}_{uv} &= \Gamma_{12}^1\mathbf{r}_u + \Gamma_{12}^2\mathbf{r}_v + B\widehat{\mathbf{N}}\\ \mathbf{r}_{vv} &= \Gamma_{22}^1\mathbf{r}_u + \Gamma_{22}^2\mathbf{r}_v + C\widehat{\mathbf{N}}\end{aligned}$$

for some scalar fields Γ_{11}^1, Γ_{11}^2, A, Γ_{12}^1, Γ_{12}^2, B, Γ_{22}^1, Γ_{22}^2, and C on D. Now, multiply these three equations by $\widehat{\mathbf{N}}$ (using the inner product on $\mathbb{R}^3$). This yields

$$\mathbf{r}_{uu}\cdot\widehat{\mathbf{N}} = A, \qquad \mathbf{r}_{uv}\cdot\widehat{\mathbf{N}} = B, \qquad \mathbf{r}_{vv}\cdot\widehat{\mathbf{N}} = C.$$

Thus, $A = L$, $B = M$, and $C = N$, where L, M, and N are the coefficients of the second fundamental form of $\mathbf{r}$. If you multiply the first equation in (↑) by $\mathbf{r}_u$ instead you obtain

$$\mathbf{r}_{uu}\cdot\mathbf{r}_u = \Gamma_{11}^1E + \Gamma_{11}^2F.$$

But

$$\mathbf{r}_{uu}\cdot\mathbf{r}_u = \frac{1}{2}(\mathbf{r}_{uu}\cdot\mathbf{r}_u + \mathbf{r}_u\cdot\mathbf{r}_{uu}) = \frac{1}{2}\frac{\partial}{\partial u}(\mathbf{r}_u\cdot\mathbf{r}_u) = \frac{1}{2}E_u$$

and so

$$\Gamma_{11}^1E + \Gamma_{11}^2F = \frac{1}{2}E_u.$$

Similarly, multiplication by $\mathbf{r}_v$ yields

$$\mathbf{r}_{uu}\cdot\mathbf{r}_v = \Gamma_{11}^1F + \Gamma_{11}^2G$$

where

$$\begin{aligned}\mathbf{r}_{uu}\cdot\mathbf{r}_v = \mathbf{r}_{uu}\cdot\mathbf{r}_v + \mathbf{r}_u\cdot\mathbf{r}_{uv} - \mathbf{r}_u\cdot\mathbf{r}_{uv} = \frac{\partial}{\partial u}(\mathbf{r}_u\cdot\mathbf{r}_v) - \frac{1}{2}(\mathbf{r}_{uv}\cdot\mathbf{r}_u + \mathbf{r}_u\cdot\mathbf{r}_{uv}) =\\ = \frac{\partial}{\partial u}(\mathbf{r}_u\cdot\mathbf{r}_v) - \frac{1}{2}\frac{\partial}{\partial v}(\mathbf{r}_u\cdot\mathbf{r}_u) = F_u - \frac{1}{2}E_v.\end{aligned}$$

Thus

$$\Gamma_{11}^1F + \Gamma_{11}^2G = F_u - \frac{1}{2}E_v$$

and you can easily solve for the Γ's in terms of the coefficients of the first fundamental form. Proceeding in the same manner with the two remaining equations in (↑) we end up with

Proposition NN

Let $\mathbf{r}\colon D \to \mathbb{R}^3$ be an f-surface with coordinates $(u, v) \in D$. Then

$$\begin{aligned}\mathbf{r}_{uu} &= \Gamma_{11}^1\mathbf{r}_u + \Gamma_{11}^2\mathbf{r}_v + L\widehat{\mathbf{N}}\\ \mathbf{r}_{uv} &= \Gamma_{12}^1\mathbf{r}_u + \Gamma_{12}^2\mathbf{r}_v + M\widehat{\mathbf{N}}\\ \mathbf{r}_{vv} &= \Gamma_{22}^1\mathbf{r}_u + \Gamma_{22}^2\mathbf{r}_v + N\widehat{\mathbf{N}}\end{aligned}$$

where

$$\Gamma^1_{11} \coloneqq \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \qquad \Gamma^2_{11} \coloneqq \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \qquad \Gamma^1_{12} \coloneqq \frac{GE_v - FG_u}{2(EG - F^2)},$$
$$\Gamma^2_{12} \coloneqq \frac{EG_u - FE_v}{2(EG - F^2)}, \qquad \Gamma^1_{22} \coloneqq \frac{2GF_v - GG_u - FG_u}{2(EG - F^2)},$$
$$\Gamma^2_{22} \coloneqq \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}$$

where E, F, and G are the coefficients of the first fundamental form of $\mathbf{r}$ and L, M, and N are the coefficients of the second fundamental form.

Definition NN

The symbols with base 'Γ' in Proposition NN are called the *Christoffel symbols* of $\mathbf{r}$, and are often written in the following matrix form:

$$\Gamma^1 \coloneqq \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} \\ \Gamma^1_{21} & \Gamma^1_{22} \end{pmatrix}, \qquad \Gamma^2 \coloneqq \begin{pmatrix} \Gamma^2_{11} & \Gamma^2_{12} \\ \Gamma^2_{21} & \Gamma^2_{22} \end{pmatrix}$$

where $\Gamma^1_{21} \coloneqq \Gamma^1_{12}$ and $\Gamma^2_{21} \coloneqq \Gamma^2_{12}$.

Notice in particular that the Christoffel symbols of an f-surface are completely determined by the first fundamental form of the f-surface.

4.3.7.2 The Covariant Derivatives of the Parameter Curve Tangent Vectors

Notice also that the above result immediately gives the covariant derivatives of $\mathbf{r}_u$ and $\mathbf{r}_v$ along the parameter curves of the f-surface. Indeed, such an f-curve

$$\mathbf{q}\colon I \to D$$

is either given by

$$\mathbf{q}(t) = \big(u(t), v(t)\big) = (t, v_0)$$

for some fixed $v_0 \in \mathbb{R}$ or by

$$\mathbf{q}(t) = \big(u(t), v(t)\big) = (u_0, t)$$

for some fixed $u_0 \in \mathbb{R}$. Without loss of generality, we will investigate only the former in detail (the analysis of the latter case is, of course, completely analogous). Then the corresponding f-curve

$$\mathbf{p} \coloneqq \mathbf{r} \circ \mathbf{q}\colon I \to \Sigma$$

on the actual surface $\Sigma = \mathbf{r}(D)$ is

$$\mathbf{p}(t) = \mathbf{r}\big(\mathbf{q}(t)\big) = \mathbf{r}(t, v_0)$$

and, at every point $t \in I$, we have a vector

$$\mathbf{r}_u(t) \coloneqq \mathbf{r}_u\big(\mathbf{q}(t)\big).$$

Thus, we might very well compute the covariant derivative of $\mathbf{r}_u(t)$ since it is a surface-tangential vector given along a curve with parameter $t = u$. In fact,

$$\frac{\nabla \mathbf{r}_u}{du} = \pi_{\mathbf{p}(t)}(\mathbf{r}_{uu}) = \pi_{\mathbf{p}(t)}\left(\Gamma_{11}^1 \mathbf{r}_u + \Gamma_{11}^2 \mathbf{r}_v + L\hat{\mathbf{N}}\right) = \Gamma_{11}^1 \mathbf{r}_u + \Gamma_{11}^2 \mathbf{r}_v.$$

If we instead had considered the rate of change of $\mathbf{r}_v$ along an f-curve $\mathbf{q}(t) = (u_0, t)$, we would have found

$$\frac{\nabla \mathbf{r}_v}{dv} = \pi_{\mathbf{p}(t)}(\mathbf{r}_{vv}) = \pi_{\mathbf{p}(t)}\left(\Gamma_{22}^1 \mathbf{r}_u + \Gamma_{22}^2 \mathbf{r}_v + N\hat{\mathbf{N}}\right) = \Gamma_{22}^1 \mathbf{r}_u + \Gamma_{22}^2 \mathbf{r}_v.$$

Of course, we can also consider the rate of change of $\mathbf{r}_v$ along an f-curve $\mathbf{q}(t) = (t, v_0)$:

$$\frac{\nabla \mathbf{r}_v}{du} = \pi_{\mathbf{p}(t)}(\mathbf{r}_{vu}) = \pi_{\mathbf{p}(t)}\left(\Gamma_{12}^1 \mathbf{r}_u + \Gamma_{12}^2 \mathbf{r}_v + M\hat{\mathbf{N}}\right) = \Gamma_{12}^1 \mathbf{r}_u + \Gamma_{12}^2 \mathbf{r}_v.$$

Finally, since $\mathbf{r}_{uv} = \mathbf{r}_{vu}$,

$$\frac{\nabla \mathbf{r}_u}{dv} = \frac{\nabla \mathbf{r}_v}{du}.$$

4.3.7.3 Vector Fields on a Surface

Definition NN

Consider a surface $\Sigma = \mathbf{r}(D) \subset \mathbb{R}^3$. A function $\mathbf{u}: D \to \mathbb{R}^3$ is called a vector field *on* Σ iff $\mathbf{u}(u, v) \in T\Sigma_{\mathbf{r}(u,v)}$ for all $(u, v) \in D$.

Below a vector field on a part of a sphere is shown.

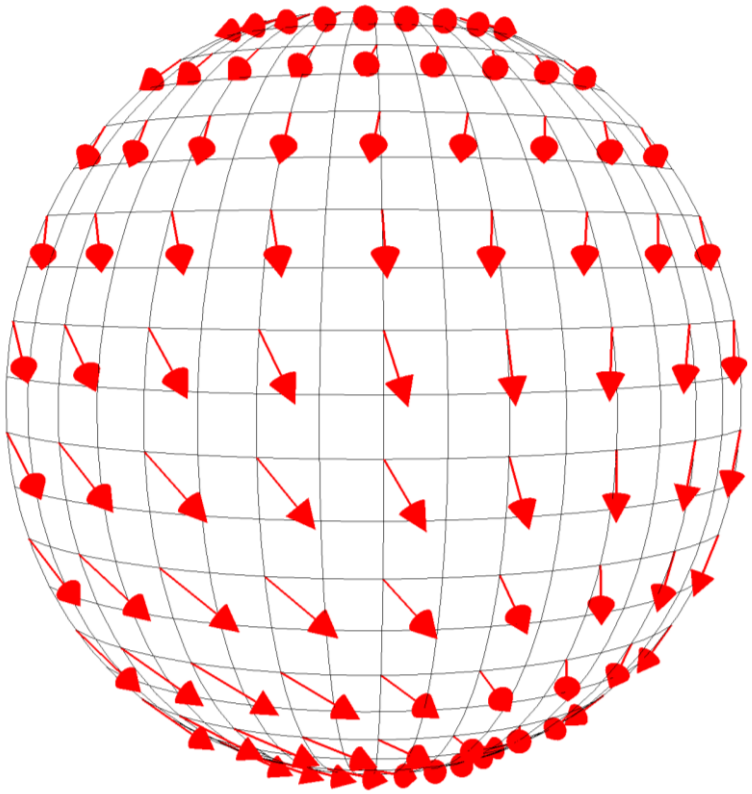


Figure 45. A vector field *on* a part of a sphere.

A general vector field on a surface Σ may be written

$$\mathbf{u}(u,v) = \alpha(u,v)\mathbf{r}_u(u,v) + \beta(u,v)\mathbf{r}_v(u,v)$$

which is automatically a field *on* Σ since $\mathbf{r}_u$ and $\mathbf{r}_v$ span $T\Sigma_{\mathbf{x}}$ at every point $\mathbf{x} = \mathbf{r}(u,v) \subset \Sigma$. Now, consider any curve $\gamma = \mathbf{p}(I) = \mathbf{r}\big(\mathbf{q}(I)\big) \subset \Sigma$, and consider, in particular, the covariant derivative of $\mathbf{u}$ along the curve. [That is, we consider the function $\mathbf{u}(t) \coloneqq \mathbf{u}\big(\mathbf{q}(t)\big)$.] This is

$$\frac{\nabla \mathbf{u}}{dt} = \frac{\nabla}{dt}(\alpha\mathbf{r}_u + \beta\mathbf{r}_v) = \frac{d\alpha}{dt}\mathbf{r}_u + \alpha\frac{\nabla\mathbf{r}_u}{dt} + \frac{d\beta}{dt}\mathbf{r}_v + \beta\frac{\nabla\mathbf{r}_v}{dt}.$$

If, in particular, the curve is a parameter curve with parameter u, then

$$\frac{\nabla \mathbf{u}}{du} = \frac{d\alpha}{du}\mathbf{r}_u + \alpha\frac{\nabla\mathbf{r}_u}{du} + \frac{d\beta}{du}\mathbf{r}_v + \beta\frac{\nabla\mathbf{r}_v}{du} = \frac{d\alpha}{du}\mathbf{r}_u + \alpha(\Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v) + \frac{d\beta}{du}\mathbf{r}_v + \beta(\Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v) =$$
$$= \left(\frac{d\alpha}{du} + \alpha\Gamma^1_{11} + \beta\Gamma^1_{12}\right)\mathbf{r}_u + \left(\frac{d\beta}{du} + \alpha\Gamma^2_{11} + \beta\Gamma^2_{12}\right)\mathbf{r}_v.$$

Similarly, we find

$$\frac{\nabla \mathbf{u}}{dv} = \frac{d\alpha}{dv}\mathbf{r}_u + \alpha\frac{\nabla\mathbf{r}_u}{dv} + \frac{d\beta}{dv}\mathbf{r}_v + \beta\frac{\nabla\mathbf{r}_v}{dv} = \frac{d\alpha}{dv}\mathbf{r}_u + \alpha(\Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v) + \frac{d\beta}{dv}\mathbf{r}_v + \beta(\Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v) =$$
$$= \left(\frac{d\alpha}{dv} + \alpha\Gamma^1_{12} + \beta\Gamma^1_{22}\right)\mathbf{r}_u + \left(\frac{d\beta}{dv} + \alpha\Gamma^2_{12} + \beta\Gamma^2_{22}\right)\mathbf{r}_v.$$

4.3.7.4 A 'Directional' Derivative

So far we have only considered the covariant derivative of a vector defined along a curve. In this section, we will introduce the rather immediate generalisation to the covariant derivative of a vector field on the (entire) surface *in the direction of another vector*. More precisely,

Definition NN

Let $\Sigma = \mathbf{r}(D)$ be a surface with parameters $(u,v) \in D$ and let $\mathbf{u}\colon D \to \mathbb{R}^3$ be a vector field on Σ. Fix some point $\mathbf{x} = \mathbf{r}(\boldsymbol{\xi}) \in \Sigma$ and let $\mathbf{v} \in D$ be a tangent vector in the parameter plane. Then the covariant derivative of $\mathbf{u}$ with respect to $\mathbf{v} \in D$ at $\boldsymbol{\xi} \in D$ is the vector

$$\nabla_{\mathbf{v}}\mathbf{u} \coloneqq \frac{\nabla \mathbf{u}'}{dt}$$

where the covariant derivative $\nabla\mathbf{u}'/dt$ is evaluated along an f-curve $\mathbf{p} \coloneqq \mathbf{r} \circ \mathbf{q}\colon I \to \Sigma$ passing through $\mathbf{x} = \mathbf{r}(\boldsymbol{\xi})$ such that the parameter plane tangent vector $\dot{\mathbf{q}} = \mathbf{v}$ at $\boldsymbol{\xi}$ and where the restriction $\mathbf{u}'(t) \coloneqq \mathbf{u}\big(\mathbf{q}(t)\big)$ of $\mathbf{u}$ to $\mathbf{q}(I)$ assigns a Σ-surface-tangent vector to each point on $\gamma \coloneqq \mathbf{r}\big(\mathbf{q}(I)\big)$.

(Notice that $\mathbf{u}\colon D \to \mathbb{R}^3$ while $\mathbf{u}'\colon I \to \mathbb{R}^3$ is defined by $\mathbf{u}' = \mathbf{u} \circ \mathbf{q}$.) Thus,

$$\nabla_{\mathbf{v}}\mathbf{u} \coloneqq \frac{\nabla \mathbf{u}'}{dt} = \pi_{\mathbf{x}}\big(\dot{\mathbf{u}}'(t)\big) = \dot{\mathbf{u}}' - \big(\hat{\mathbf{N}} \cdot \dot{\mathbf{u}}'\big)\hat{\mathbf{N}}$$

where

$$\dot{\mathbf{u}}'(t) = \frac{d}{dt}\Big(\mathbf{u}\big(\mathbf{q}(t)\big)\Big) = \dot{\mathbf{u}}\big(\mathbf{q}(t)\big) \cdot \dot{\mathbf{q}}(t) = \dot{\mathbf{u}} \cdot \mathbf{v}$$

where, of course, $\dot{\mathbf{u}}$ is a 3×2 Jacobian matrix. Therefore,

$$\nabla_{\mathbf{v}}\mathbf{u} = \dot{\mathbf{u}}\cdot\mathbf{v} - \left(\widehat{\mathbf{N}}\cdot(\dot{\mathbf{u}}\cdot\mathbf{v})\right)\widehat{\mathbf{N}}$$

The last formula shows that the covariant derivative $\nabla_{\mathbf{v}}\mathbf{u}$ is well-defined, that is, independent of the particular choice of f-curve $\mathbf{q}$ mentioned in the definition; you only need its tangent vector $\mathbf{v}$ at the point $\boldsymbol{\xi}$ of evaluation.

Lemma NN

Let $\Sigma = \mathbf{r}(D)$ be a surface with parameters $(u,v)\in D$ and let $\mathbf{u}\colon D\to\mathbb{R}^3$ be a vector field on Σ. Fix a point $\mathbf{x} = \mathbf{r}(\boldsymbol{\xi})$ and let $\mathbf{v} = (\chi,\psi)\in D$. Then

$$\nabla_{\mathbf{v}}\mathbf{u} = \chi\frac{\nabla\mathbf{u}}{du} + \psi\frac{\nabla\mathbf{u}}{dv},$$

where, as indicated, ∇/du and ∇/dv are covariant derivatives along parameter curves with varying u and v, respectively.

Proof

Let $\gamma = \mathbf{p}(I) = \mathbf{r}\big(\mathbf{q}(I)\big)$ be a curve on Σ with parameter-plane tangent $\dot{\mathbf{q}}(t) = \mathbf{v}$ at $\mathbf{q}(t) = \boldsymbol{\xi}$. Then, by definition,

$$\nabla_{\mathbf{v}}\mathbf{u} = \frac{\nabla\mathbf{u}'}{dt} = \pi_{\mathbf{p}(t)}\left(\frac{d\mathbf{u}'}{dt}\right) = \pi_{\mathbf{p}(t)}\left(\frac{d}{dt}\big(\mathbf{u}\big(\mathbf{q}(t)\big)\big)\right) = \pi_{\mathbf{p}(t)}\left(\dot{\mathbf{u}}\big(\mathbf{q}(t)\big)\cdot\dot{\mathbf{q}}(t)\right) = \pi_{\mathbf{p}(t)}\big(\dot{\mathbf{u}}\big(\mathbf{q}(t)\big)\cdot\mathbf{v}\big) =$$
$$= \pi_{\mathbf{p}(t)}(\mathbf{u}_u\xi + \mathbf{u}_v\psi) = \xi\pi_{\mathbf{p}(t)}(\mathbf{u}_u) + \psi\pi_{\mathbf{p}(t)}(\mathbf{u}_v) = \chi\frac{\nabla\mathbf{u}}{du} + \psi\frac{\nabla\mathbf{u}}{dv}$$

at $\boldsymbol{\xi}$, where $\mathbf{u}' \coloneqq \mathbf{u}\circ\mathbf{q}\colon I\to\mathbb{R}^3$ as usual. ■

Putting things together, we finally end up with

Theorem NN

Let $\Sigma = \mathbf{r}(D)$ be a surface with parameters $(u,v)\in D$ and let $\mathbf{u}\colon D\to\mathbb{R}^3$ be a vector field on Σ. If $\mathbf{u}(u,v) = \alpha(u,v)\mathbf{r}_u + \beta(u,v)\mathbf{r}_v$ and $\mathbf{v} = (\chi,\psi)\in D$ then

$$\nabla_{\mathbf{v}}\mathbf{u} = \left(\chi\frac{d\alpha}{du} + \chi\alpha\Gamma_{11}^1 + \chi\beta\Gamma_{12}^1 + \psi\frac{d\alpha}{dv} + \psi\alpha\Gamma_{12}^1 + \psi\beta\Gamma_{22}^1\right)\mathbf{r}_u +$$
$$+ \left(\chi\frac{d\beta}{du} + \chi\alpha\Gamma_{11}^2 + \chi\beta\Gamma_{12}^2 + \psi\frac{d\beta}{dv} + \psi\alpha\Gamma_{12}^2 + \psi\beta\Gamma_{22}^2\right)\mathbf{r}_v.$$

Proof

Combining Proposition NN and the results of Section NN, we find

$$\nabla_{\mathbf{v}}\mathbf{u} = \chi\frac{\nabla\mathbf{u}}{du} + \psi\frac{\nabla\mathbf{u}}{dv} =$$
$$= \chi\left(\left(\frac{d\alpha}{du} + \alpha\Gamma_{11}^1 + \beta\Gamma_{12}^1\right)\mathbf{r}_u + \left(\frac{d\beta}{du} + \alpha\Gamma_{11}^2 + \beta\Gamma_{12}^2\right)\mathbf{r}_v\right) +$$
$$+ \psi\left(\left(\frac{d\alpha}{dv} + \alpha\Gamma_{21}^1 + \beta\Gamma_{22}^1\right)\mathbf{r}_u + \left(\frac{d\beta}{dv} + \alpha\Gamma_{21}^2 + \beta\Gamma_{22}^2\right)\mathbf{r}_v\right)$$

which simplifies to the desired result. ■

4.3.7.5 The Significance of the Result

The reader might wonder why we have spent so much effort in deriving such a seemingly horrid result as the one of Theorem NN. The point of Theorem NN is that it gives the covariant derivative of a vector field on a surface (and, in any given direction), without the need to know anything about the particular embedding of the surface in $\mathbb{R}^3$. Indeed, you only need to know a coordinate system on the surface, and its first fundamental form. This gives you the Christoffel symbols Γ^{ijk}, and then you specify your vector field in terms of the basis and at any point, and you specify your direction in terms of the parameter plane alone. This approach will be crucial in the next chapter where we consider manifolds (such as surfaces) that are not (explicitly) embedded in some higher-dimensional space. In that chapter, we will also reformulate much of our theory in terms of tensors, and then the result (↑) will look much simpler thanks to tensor notation and a notational device due to Einstein.

4.3.8 The Covariant Derivative, Parallel Transport, and Geodesics

In this final section of this chapter, we will use the concept of the covariant derivative to define the concept of parallel transport and we will also find a new approach to the study of surface geodesics.

4.3.8.1 Parallel Transport

In $\mathbb{R}^n$, the *tangent space* $T\mathbb{R}^n_{\mathbf{x}}$ at $\mathbf{x} \in \mathbb{R}^n$ is a copy of $\mathbb{R}^n$ with at $\mathbf{x} \in \mathbb{R}^n$ and with the same geometric basis vectors, as exemplified below in the case of $n = 2$, where the radius vector of $\mathbf{x}$ is shown in orange:

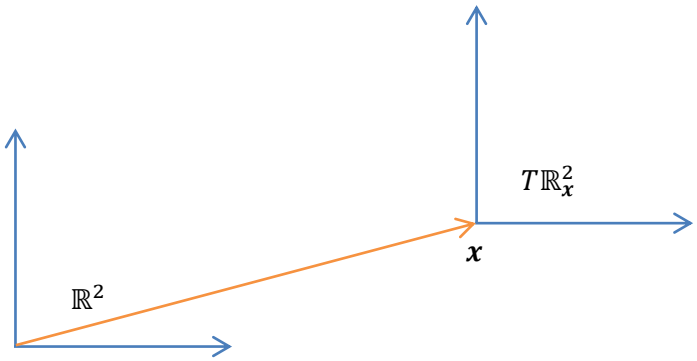


Figure 46. Every tangent space of $\mathbb{R}^n$ is a copy of $\mathbb{R}^n$ with the same basis vectors.

Since for every $\mathbf{x} \in \mathbb{R}^n$, the tangent space $T\mathbb{R}^n_{\mathbf{x}} = \mathbb{R}^n$, you seldom talk explicitly about tangent spaces of $\mathbb{R}^n$. Indeed, we have had an entire chapter on classical mechanics without even mentioning it, although, for instance, the velocity vector of a particle at $\mathbf{x} \in \mathbb{R}^3$, strictly speaking, is a vector $\mathbf{v} \in T\mathbb{R}^3_{\mathbf{x}}$ in the tangent space of $\mathbb{R}^3$ at $\mathbf{x}$. If such a particle is moving with constant velocity, then we associate a velocity vector $\mathbf{v}_{\mathbf{x}}$ to each tangent space $T\mathbb{R}^3_{\mathbf{x}}$, and all these vectors are, geometrically, the same, and they also have the same components in each tangent space. Thus we say that the vector $\mathbf{v} \coloneqq \mathbf{v}_{\mathbf{x}}$ for some initial $\mathbf{x} \in \mathbb{R}^3$ is *parallel propagated* (or *transported*). More generally, if there is an assignment of a vector $\mathbf{v} \in T\mathbb{R}^n_{\mathbf{x}}$ to each point $\mathbf{x} \in \gamma$ of a curve $\gamma \subset \mathbb{R}^n$ such that the components of $\mathbf{v}$ are the same in every tangent space, then $\mathbf{v}$ is said to be parallel propagated (or transported) along γ, as exemplified below.

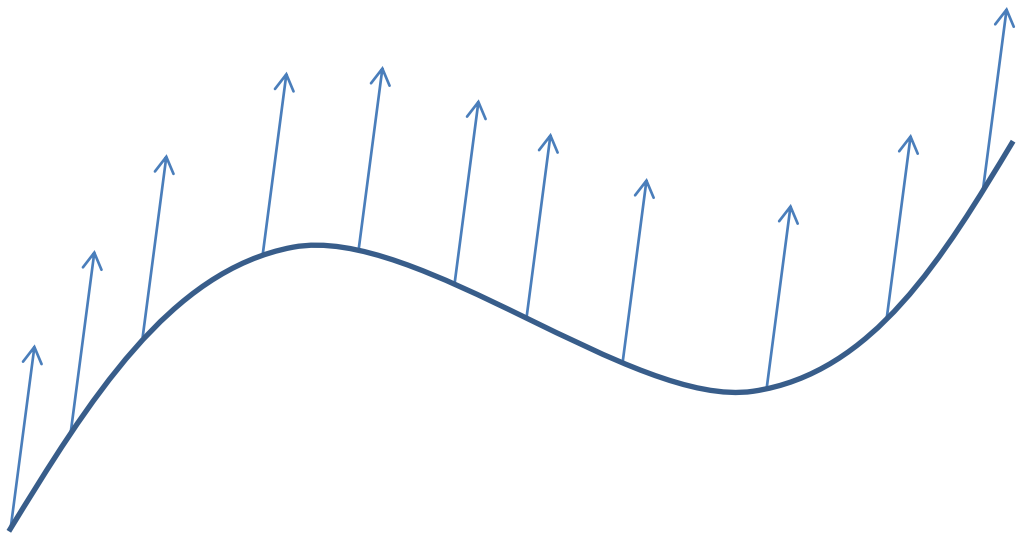


Figure 47. A vector is parallel propagated along a curve in $\mathbb{R}^2$.

Consider now a (manifold) surface $\Sigma \subset \mathbb{R}^3$. What should we mean by a vector being parallel propagated along a curve $\gamma \subset \Sigma$? Clearly, since $\Sigma \subset \mathbb{R}^3$, $\gamma \subset \mathbb{R}^3$ and we could simply mean the usual thing considering the vector as a vector in $\mathbb{R}^3$ and the curve as a curve in $\mathbb{R}^3$. However, we want to find a concept that is meaningful for a two-dimensional being living *inside* the surface, not being aware of the ambient space. For example, consider your flagpole-related paradox, and your intense argument with Buggy. According to Buggy, the flagpole (your velocity vector, remember?) is certainly not parallel propagated when you stroll along the equator, even though, to you, it appears to be. It thus makes sense to make

Definition NN

Let $\gamma = \mathbf{p}(I) = \mathbf{r}\big(\mathbf{q}(I)\big)$, where $I \coloneqq [a, b]$ be a curve on a surface $\Sigma = \mathbf{r}(D)$. Let $\mathbf{u}\colon I \to \mathbb{R}^3$ where $\mathbf{u}(t) \in T\Sigma_{\mathbf{p}(t)}$ for every $t \in I$. Then the vector $\mathbf{u}(a)$ is *parallel propagated* (or *transported*) along γ iff $\nabla\mathbf{u}/dt \equiv 0$.

Consider Example NN again, where you are walking along the equator. Since $\nabla\dot{\mathbf{p}}/dt \equiv 0$, the flagpole (your tangent vector) *is* parallel propagated along the equator.

4.3.8.2 Geodesics

The concept of parallel propagation is defined for every surface-tangential vector defined along a curve on the surface. Nevertheless, it is of *particular* interest when the vector is the velocity (or tangent) vector of the curve.

Indeed, we have (which shouldn't be a major surprise to the reader)

Theorem NN

A unit-speed curve on a surface is a geodesic if and only if its tangent vector is parallel propagated along the curve.

Proof

Let $\gamma = \mathbf{p}(I) = \mathbf{r}\big(\mathbf{q}(I)\big)$ be a curve on a surface $\Sigma = \mathbf{r}(D)$, where $\mathbf{p}$ is unit-speed. Then the covariant derivative of the tangent vector is

$$\frac{\nabla\dot{\mathbf{p}}}{dt} = \pi_{\mathbf{p}(t)}(\ddot{\mathbf{p}}) = \pi_{\mathbf{p}(t)}\left(\kappa_n\hat{\mathbf{N}} + \kappa_g\hat{\mathbf{N}}\times\dot{\mathbf{p}}\right) = \pi_{\mathbf{p}(t)}\left(\kappa_n\hat{\mathbf{N}}\right) + \pi_{\mathbf{p}(t)}\left(\kappa_g\hat{\mathbf{N}}\times\dot{\mathbf{p}}\right) = \pi_{\mathbf{p}(t)}\left(\kappa_g\hat{\mathbf{N}}\times\dot{\mathbf{p}}\right) =$$
$$= \kappa_g\pi_{\mathbf{p}(t)}\left(\hat{\mathbf{N}}\times\dot{\mathbf{p}}\right).$$

But $\hat{\mathbf{N}} \neq \mathbf{0}$ $\dot{\mathbf{p}} \neq \mathbf{0}$, and $\hat{\mathbf{N}} \perp \dot{\mathbf{p}}$. Therefore, $\hat{\mathbf{N}}\times\dot{\mathbf{p}} \neq \mathbf{0}$. In addition, $\left(\hat{\mathbf{N}}\times\dot{\mathbf{p}}\right) \perp \hat{\mathbf{N}}$, and so $\hat{\mathbf{N}}\times\dot{\mathbf{p}} \in T\Sigma_{\mathbf{p}(t)}$. Thus, $\pi_{\mathbf{p}(t)}\left(\hat{\mathbf{N}}\times\dot{\mathbf{p}}\right) \neq \mathbf{0}$. Therefore,

$$\frac{\nabla\dot{\mathbf{p}}}{dt} = 0 \Leftrightarrow \kappa_g = 0,$$

that is, the tangent vector $\dot{\mathbf{p}}$ is parallel propagated along the curve if and only if the curve is a geodesic. ■

5 Manifolds and Tensors

This chapter is devoted to the introduction of the modern language of differential geometry. We will begin with a (fairly) simple introduction to tensors; following that, we will introduce the manifold, and, finally, we will introduce the concept of tensor fields on manifolds.

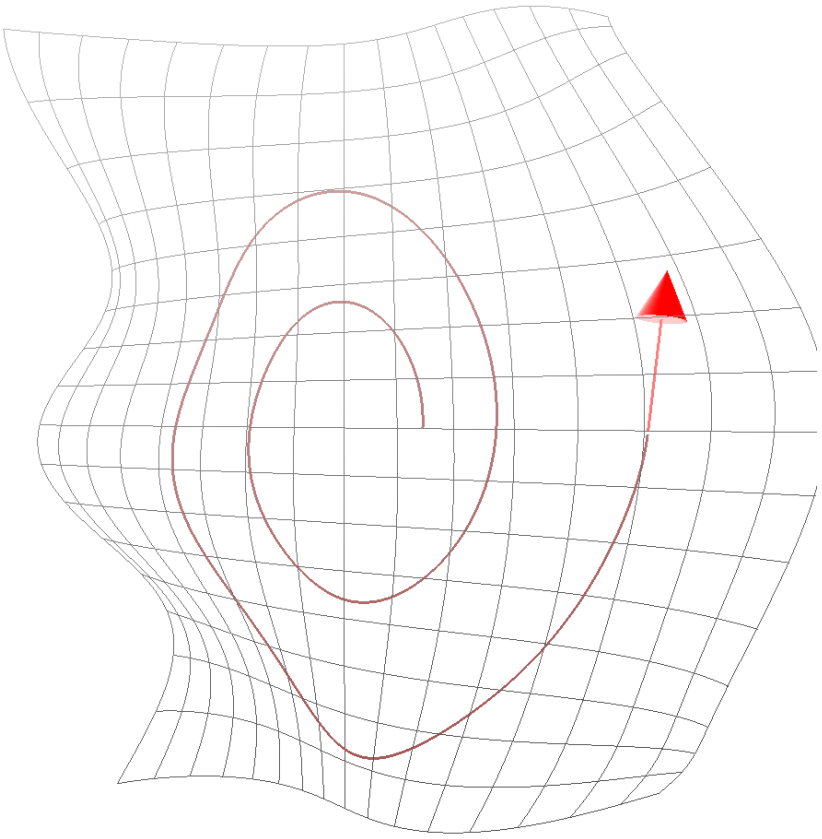


Figure 48. A curve on a surface, with a tangent vector on it.

5.1 Tensors

General relativity (among other fields of physics) is based on tensor algebra and tensor analysis; unfortunately, this field of mathematics is very often introduced inadequately in introductory text on (and even courses in) relativity theory. The reason, I believe, is twofold. For one thing, tensor calculus *is* not the simplest branch of mathematics there is. In particular, there are many (seemingly different) approaches to tensor analysis in different fields of mathematics and physics. The notation used in tensor analysis is also rather 'intricate', and notational conventions (that appears to be unrigorous[52]) are used everywhere. However, I do believe that the major problem lies in the failure to recognize that one *first* has to study mathematics, *before* one can study physics. Indeed, many (probably even 'most') introductory texts on general relativity assumes no previous knowledge of tensor analysis, and introduces the concepts on the fly, or in a far too brief chapter that completely lacks any mathematical insight into the subject.

It is in order to remedy this problem, that the last chapter was devoted entirely to the classical differential geometry of curves and surfaces in Euclidean $\mathbb{R}^3$, and that this chapter is devoted to the concepts of modern differential geometry of manifolds, using tensor notation.

5.1.1 Some Concepts from Linear Algebra

We will review some basic concepts from linear algebra that are essential for our development of tensor algebra. In this entire chapter, we write simply 'vector space' to denote *a finite-dimensional vector space over the real numbers*; in addition, since every real n-dimensional vector space with a prescribed basis is isomorphic to $\mathbb{R}^n$, you can (almost) always think of this space.

5.1.1.1 Multilinear Maps

Let X be a vector space. A real-valued function $f: X \to \mathbb{R}$ is said to be *linear* iff

$$f(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) = \alpha_1 f(\mathbf{v}_1) + \alpha_2 f(\mathbf{v}_2)$$

for all $\mathbf{v}_1$ and $\mathbf{v}_2 \in X$ and all α_1 and $\alpha_2 \in \mathbb{R}$. A real-valued function

$$f: \underbrace{X \times \cdots \times X}_{k \text{ factors}} \to \mathbb{R}$$

is called *multilinear* iff, for every $i = 1, \dots, k$, the function $g_i: X \to \mathbb{R}$ is linear where $g_i(\mathbf{v}) \coloneqq f(\dots, \mathbf{v}, \dots)$, where $\mathbf{v} \in X$ is the ith argument of f, and the other arguments of f are arbitrary fixed elements of X. In particular, if $k = 2$, f is said to be *bilinear*. More generally, a real-valued function $f: X_1 \times X_2 \times \cdots \times X_k \to \mathbb{R}$, where each X_k is a vector space, is said to be multilinear (or bilinear if $k = 2$) iff for every $i = 1, \dots, k$ the function $g_i: X_i \to \mathbb{R}$ defined by $g_i(\mathbf{v}) \coloneqq f(\dots, \mathbf{v}, \dots)$, with the $\mathbf{v} \in X_i$ at the ith argument, is linear (while all other arguments are arbitrary and fixed).

5.1.1.2 The Inner Product

Proposition NN

Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space X of dimension n. Then, given a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of X, there exists a symmetric, positive-definite, $n \times n$ matrix A such that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}, \qquad \forall \mathbf{u}, \mathbf{v} \in X.$$

[52] For instance, is v^a a vector or a single component of a vector (that is, a number)? Is T_{ab} a multilinear map (that is, a function), a matrix, or a single number? In what sense is g^{ab} the 'inverse' of g_{ab}?

In fact, $A_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$. Conversely, every symmetric, positive-definite, $n \times n$ matrix A induces an inner product, that is, the function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle \coloneqq \mathbf{u}^T A \mathbf{v}$$

satisfies the axioms of an inner product.

Proof

Let $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$. Then, by the axioms of an inner product,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + \cdots + u_n \mathbf{e}_n, \quad v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \cdots + v_n \mathbf{e}_n \rangle =$$
$$= \sum_{i=1}^{n} u_i \langle \mathbf{e}_i, \quad v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \cdots + v_n \mathbf{e}_n \rangle = \sum_{i=1}^{n} \left(u_i \sum_{j=1}^{n} v_j \langle \mathbf{e}_i, \mathbf{e}_j \rangle \right) = \mathbf{u}^T A \mathbf{v}$$

where the (i, j)th element of the $n \times n$ matrix A is $\langle \mathbf{e}_i, \mathbf{e}_j \rangle$. Since $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle \mathbf{e}_j, \mathbf{e}_i \rangle$, A is symmetric. In addition, since $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ with equality iff $\mathbf{v} = \mathbf{0}$, the matrix A is positive definite. Conversely, if $\langle \mathbf{u}, \mathbf{v} \rangle \coloneqq \mathbf{u}^T A \mathbf{v}$ where A is a symmetric, positive-definite $n \times n$ matrix, it is easy to see that the function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ satisfies the axioms of an inner product. ■

In elementary linear algebra one almost exclusively work with an orthonormal basis, in which case A is the unit matrix; that is, $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = [i = j]$ where $[\cdot]$ is the Iverson bracket.

5.1.1.3 Linear Functionals

A *functional* on X is a function $f : X \to \mathbb{R}$. Let f and g be *linear* functionals on X. Then $(f + g) : X \to \mathbb{R}$ defined by $(f + g)(\mathbf{v}) \coloneqq f(\mathbf{v}) + g(\mathbf{v}), \forall \mathbf{v} \in X$ is also a linear functional on X, as is, for every scalar $\alpha \in \mathbb{R}$, the function $\alpha f : X \to \mathbb{R}$ defined by $(\alpha f)(\mathbf{v}) \coloneqq \alpha f(\mathbf{v}), \forall \mathbf{v} \in X$. Therefore, the set of all linear functionals on X is also a vector space, called the *dual space* of X, denoted X^*. An element $v \in X^*$ is called a *covector*.

Given a basis $\mathbf{e}_1$, ..., $\mathbf{e}_n$ of X, there is a natural basis of X^*, called the *dual basis*, consisting of the covectors $\boldsymbol{\sigma}_1$, ..., $\boldsymbol{\sigma}_n$ defined by

$$\boldsymbol{\sigma}_k(\mathbf{v}) = \boldsymbol{\sigma}_k(v_1 \mathbf{e}_1 + \cdots + v_n \mathbf{e}_n) \coloneqq v_k, \qquad \forall \mathbf{v} \in X;$$

that is, $\boldsymbol{\sigma}_k$ reads off the kth component of its argument vector. We remark that an equivalent definition is

$$\boldsymbol{\sigma}_i(\mathbf{e}_j) \coloneqq [i = j].$$

Proposition NN

The covectors $\boldsymbol{\sigma}_1$, ..., $\boldsymbol{\sigma}_n$ defined above constitute a basis of X^*.

Proof

Let $f \in X^*$ be any covector. Then, by linearity, its image on $\mathbf{v} = v_1 \mathbf{e}_1 + \cdots + v_n \mathbf{e}_n \in X$ is

$$f(\mathbf{v}) = f(v_1 \mathbf{e}_1 + \cdots + v_n \mathbf{e}_n) = v_1 f(\mathbf{e}_1) + \cdots + v_n f(\mathbf{e}_n).$$

On the other hand, we want to find unique scalars λ_1, ..., λ_n such that

$$f = \lambda_1 \boldsymbol{\sigma}_1 + \cdots + \lambda_n \boldsymbol{\sigma}_n,$$

that is, such that

$$f(\mathbf{v}) = \lambda_1 \boldsymbol{\sigma}_1(\mathbf{v}) + \cdots + \lambda_n \boldsymbol{\sigma}_n(\mathbf{v}) = \lambda_1 v_1 + \cdots + \lambda_n v_n,$$

for all $\mathbf{v} \in X$. The only possibility is then

$$\lambda_1 = f(\mathbf{e}_1), \qquad \dots, \qquad \lambda_n = f(\mathbf{e}_n).$$

Thus, every covector f can be written in the form (↑) for a unique sequence of coefficients (↑), and so $\boldsymbol{\sigma}_1$, ..., $\boldsymbol{\sigma}_n$ is a basis of X^*. ■

5.1.2 The Definition of a Tensor

Definition NN

Let X be a vector space. Then a *covariant tensor of rank* $k \in \mathbb{Z}^+$ is a multilinear map $f: \underbrace{X \times \cdots \times X}_{k \text{ factors}} \to \mathbb{R}$.

A *contravariant tensor of rank* $k \in \mathbb{Z}^+$ is a multilinear map $f: \underbrace{X^* \times \cdots \times X^*}_{k \text{ factors}} \to \mathbb{R}$.

A multilinear map $f: \underbrace{X^* \times \cdots \times X^*}_{k \text{ factors}} \times \underbrace{X \times \cdots \times X}_{l \text{ factors}} \to \mathbb{R}$ is called a *tensor of type* (k, l), where $k, l \in \mathbb{N}$ and the *rank* of f is $k + l \in \mathbb{Z}^+$. The tensor is *mixed* if both k and $l > 0$.

Notice that,

- a tensor of type $(0, k)$ is a covariant tensor of rank k,
- a tensor of type $(k, 0)$ is a contravariant tensor of rank k, and
- any tensor is *either* covariant, contravariant, *or* mixed.

In addition,

- a vector 'is' a contravariant tensor of rank 1 (as we will see shortly), and
- a covector is a covariant tensor of rank 1 (as is obvious).

Most importantly,

- the image of a tensor does not depend upon the choice of basis in X.

The last part means that the *image* of the tensor is *the same number* when computed using every basis. It is also convenient to define a tensor of type $(0, 0)$ (and rank 0) to be simply a number (an element of $\mathbb{R}$, to be precise). Such a number is also called a *scalar*, and is independent of basis.

Let us consider a few examples of tensors that we already know of. First of all, if there is an inner product on X, then the inner product is a covariant tensor of rank 2. Indeed, the function $f: X \times X \to \mathbb{R}$ defined by

$$f(\mathbf{u}, \mathbf{v}) \coloneqq \langle \mathbf{u}, \mathbf{v} \rangle$$

is certainly bilinear. If we fix some $\mathbf{u} \in X$, then the function $f: X \to \mathbb{R}$ defined by

$$f(\mathbf{v}) \coloneqq \langle \mathbf{u}, \mathbf{v} \rangle$$

is a covariant tensor of rank 1, that is, a covector: $f \in X^*$. Thus, any vector induces a covariant tensor of rank 1. Similarly, given a basis, any square matrix A induces a covariant tensor of rank 2 by its quadratic form; that is, $f: X \times X \to \mathbb{R}$ where

$$f(\mathbf{u},\mathbf{v}) \coloneqq \mathbf{u}^T A\mathbf{v}.$$

(If A is also symmetric and positive definite, this is actually an inner product.) In the previous chapter we defined the first and second fundamental forms, $\mathcal{F}$ and $\mathcal{M}$, that both induce covariant tensors of rank 2. Both $\mathcal{F}$ and $\mathcal{M}$ are symmetric by construction, but $\mathcal{F}$ is also positive-definite, and so the tensor it induces is in fact an inner product.

Notice also that every vector $\mathbf{v} \in X$ induces a *contravariant* tensor of rank 1, namely, $f(u) \coloneqq u(\mathbf{v})$ for every $u \in X^*$, that is, $\mathbf{v}$ lets a covector act on it. It is easy to see that f is linear. In fact, the 'pairing operation' $T\colon X^* \times X \to \mathbb{R}$ defined by $T(u,\mathbf{v}) \coloneqq u(\mathbf{v})$ for all $u \in X^*$ and $\mathbf{v} \in X$ is itself a mixed tensor of type $(1,1)$.

Finally, a *linear transformation* $F\colon X \to X$ naturally induces a mixed tensor $T\colon X^* \times X \to \mathbb{R}$ of type $(1,1)$ by

$$T(u,\mathbf{v}) \coloneqq u\big(F(\mathbf{v})\big).$$

5.1.3 The Components of a Tensor

If we have chosen a basis for X, then a tensor of any type can be described by its (finite number of) *components*, in much the same way as a linear transformation can be described by its components (that is, by its matrix). Of course, in both cases, this is an immediate effect of the severe restriction that the function is *linear*.

5.1.3.1 The Components of a Covector

Consider first a covector, that is, a covariant tensor f of rank 1. Its image on the vector $\mathbf{v} = v_1\mathbf{e}_1 + \cdots v_n\mathbf{e}_n$ is

$$f(\mathbf{v}) = f(v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n) = v_1 f(\mathbf{e}_1) + \cdots + v_n f(\mathbf{e}_n).$$

But the numbers $f(\mathbf{e}_1)$, ..., $f(\mathbf{e}_n)$, which we denote f_1, ..., f_n, do not depend upon the argument $\mathbf{v}$, and are called the *components* of f with respect to the basis $\mathbf{e}_1$, ..., $\mathbf{e}_n$. As we saw in the proof of Proposition NN, the components defined this way are also the coordinates of the covector f considered a member of X^* relative the dual basis $\boldsymbol{\sigma}_1$, ..., $\boldsymbol{\sigma}_n$; indeed,

$$f = f_1\boldsymbol{\sigma}_1 + \cdots + f_2\boldsymbol{\sigma}_n, \qquad \forall f \in X^*.$$

It is common to represent a vector as a column matrix of its components and a covector as a row matrix of its components (given a particular basis, of course):

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$f = \begin{pmatrix} f(\mathbf{e}_1) & \cdots & f(\mathbf{e}_n) \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} f_1 & \cdots & f_n \end{pmatrix}.$$

Then the image

$$f(\mathbf{v}) = f \cdot \mathbf{v} = \begin{pmatrix} f_1 & \ldots & f_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

can be represented as a simple matrix multiplication of the covector f and the vector $\mathbf{v}$, both being regarded as matrices.

5.1.3.2 The Components of a Vector

A contravariant tensor of rank 1 is, by definition, a linear map $f: X^* \to \mathbb{R}$. As we have seen, every vector $\mathbf{v} \in \mathbb{R}$ induces such a map by letting a covector act on it, that is, by

$$f(u) \coloneqq u(\mathbf{v}), \qquad \forall u \in X^*.$$

If

$$\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$$

and

$$u = u_1\boldsymbol{\sigma}_1 + \cdots + u_n\boldsymbol{\sigma}_n$$

then, by linearity,

$$f(u) = u(\mathbf{v}) = u(v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n) = v_1u(\mathbf{e}_1) + \cdots + v_nu(\mathbf{e}_n) = v_1u_1 + \cdots + u_nv_n.$$

The numbers v_1, ..., v_n do not depend upon the covector u and are called the components of the contravariant tensor f of rank 1 induced by $\mathbf{v}$. But they are also the components of the vector $\mathbf{v} \in X$. This is the reason why vectors are considered contravariant tensors of rank 1. We can write

$$f(u) = u(\mathbf{v}) = u \cdot \mathbf{v} = \begin{pmatrix} u_1 & \dots & u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

as usual for the 'marriage' between a covector and a vector.

5.1.3.3 The Components of an Arbitrary Tensor

Finally, consider an arbitrary tensor T of type (k, l), acting on the k covectors $f = f_1\boldsymbol{\sigma}_1 + \cdots + f_n\boldsymbol{\sigma}_n$, $g = g_1\boldsymbol{\sigma}_1 + \cdots + g_n\boldsymbol{\sigma}_n$, ..., and the l vectors $\mathbf{u} = u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n$, $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$, ... By linearity,

$$T(f, g, \dots, \mathbf{u}, \mathbf{v}, \dots) = \sum_{i=1}^{n} f_i T(\boldsymbol{\sigma}_i, g, \dots, \mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} f_i \sum_{j=1}^{n} g_j T\big(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j, \dots, \mathbf{u}, \mathbf{v}\big) = \cdots =$$

$$= \sum_{\substack{i,j,\dots=1 \\ a,b,\dots=1}}^{n} f_i g_j \dots u_a v_b \dots T\big(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j, \dots, \mathbf{e}_a, \mathbf{e}_b, \dots\big).$$

The n^{k+l} numbers $T\big(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j, \dots, \mathbf{e}_a, \mathbf{e}_b, \dots\big)$, which clearly depend only on the tensor T and the basis vectors $\mathbf{e}_1$, ..., $\mathbf{e}_n$ of X (and the dual basis vectors), are called the *components* of the tensor T relative to the basis $\mathbf{e}_1$, ..., $\mathbf{e}_n$ (and the dual basis). Thus, these n^{k+l} numbers completely specify the tensor (given a particular basis).

For example, consider $X = \mathbb{R}^3$ and a tensor T of rank 2. Given a basis in $\mathbb{R}^3$, T is uniquely determined by its $3^2 = 9$ components, white can be displayed in a matrix. It should be clear that a tensor of rank k requires a k-dimensional table (or 'array') of numbers in order to specify all its components (relative to a basis).

5.1.3.4 The Linear Transformation is a Tensor

We have seen how a linear transformation $F: X \to X$ naturally induces a mixed tensor $T: X^* \times X \to \mathbb{R}$ by $T(u, \mathbf{v}) \coloneqq u\big(F(\mathbf{v})\big)$. By definition, the components of T are

$$T(\boldsymbol{\sigma}_i, \mathbf{e}_j) = \boldsymbol{\sigma}_i\left(F(\mathbf{e}_j)\right) = A_{ij},$$

where A_{ij}, apparently, is the ith component of the image of the jth basis vector. But these are also the components of the matrix of F! Thus, in this sense, not only does a linear transformation *induce* a tensor of type $(1, 1)$, but a linear transformation *is* such a tensor.

5.1.3.5 The Concrete Index Notation

If T is a tensor of any type, then given a basis, it is customary and very convenient to denote the specific component $T(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j, \dots, \mathbf{e}_a, \mathbf{e}_b, \dots)$ by

$$T^{ij\cdots}_{ab\cdots} \coloneqq T(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j, \dots, \mathbf{e}_a, \mathbf{e}_b, \dots).$$

By 'specific', we mean that the indices $i, j, \dots, a, b, \dots$ are in fact placeholders for actual numbers. (That remark was probably annoyingly obvious to a reader with no previous knowledge of tensor analysis!) In particular, if T is a vector, its components will be written T^i. For example, the vector $\mathbf{v} = 5\mathbf{e}_1 + 7\mathbf{e}_2 + 9\mathbf{e}_3$ has components

$$v^1 = 5, \qquad v^2 = 7, \qquad v^3 = 9$$

and may be written[53]

$$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3.$$

On the other hand, if $u = 5\boldsymbol{\sigma}_1 + 7\boldsymbol{\sigma}_2 + 9\boldsymbol{\sigma}_3$ is a covector, then its components are written

$$u_1 = 5, \qquad u_2 = 7, \qquad u_3 = 9$$

and the covector itself may be written

$$u = u_1\boldsymbol{\sigma}^1 + u_2\boldsymbol{\sigma}^2 + u_3\boldsymbol{\sigma}^3$$

where you notice that we have moved the indices on the dual basis vectors from the bottom to the top. The (immediate) reason for this is that we may then employ the *Einstein summation convention*. Consider a general vector $\mathbf{v}$ and a general covector u. Then

$$\mathbf{v} = \sum_{i=1}^{n} v^i\mathbf{e}_i \quad \text{and} \quad u = \sum_{i=1}^{n} u_i\boldsymbol{\sigma}^i.$$

The Einstein summation convention allows us to *omit* the summation signs in these expressions, and write simply

$$\mathbf{v} = v^i\mathbf{e}_i \quad \text{and} \quad u = u_i\boldsymbol{\sigma}^i.$$

The general formulation of the Einstein summation convention is given below.

Einstein's Summation Convention

Every time there is a *repeated* index (a *letter*), *once in superscript and once in subscript*, in a single term, then summation over all possible index values with the term as summand is implied.

From now on, we will employ the summation convention, and we will not always make an explicit remark about it! That means, that if you are new to tensor analysis, you might want to read the box above, make yourself a cup of coffee, then read the box again, then go to bed and fall

[53] Before we have always used subscripts for vector indices, but from now on, we will use superscripts to distinguish vector components from covector components.

asleep, then wake up, then read the box again, then take the dog for a walk, and then read the box again. After that, you are less likely to forget the convention.

In particular, if $\mathbf{v} = v^i \mathbf{e}_i$ is a vector and $u = u_i \boldsymbol{\sigma}^i$ is a covector, we may write the pairing simply as

$$u(\mathbf{v}) = u_i v^i.$$

More generally, if T is a tensor of type (k, l), then its image is (cf. Eq. NN)

$$T(f, g, \dots, \mathbf{u}, \mathbf{v}, \dots) = \sum_{\substack{i,j,\dots=1 \\ a,b,\dots=1}}^{n} f_i g_j \dots u^a v^b \dots T^{ij\dots}_{ab\dots} = T^{ij\dots}_{ab\dots} f_i g_j \dots u^a v^b \dots.$$

5.1.3.5.1 The Order of Arguments

We should note that a multilinear function of vectors and covectors is a tensor even if the arguments are not ordered in such a way that every covector slot precede every vector slot. For example, of course

$$f: X^* \times X^* \times X \to \mathbb{R}$$

is a tensor if it is multilinear. But

$$g: X^* \times X \times X^* \to \mathbb{R}$$

is also a tensor, if it is multilinear. Since we want to allow tensors with any ordering of the arguments, we will sometimes denote the components of the tensor in such a way that there is exactly *one* index, superscript or subscript, at every index column, and the index at the ith column corresponds to the ith argument of the tensor. For example, when a clear distinction is required, the components of g will *not* be written

$$g^{ab}_c,$$

but rather

$$g^a{}_c{}^b$$

from which it is clear, for instance, that the vector argument is the *second* argument of g. In addition, we need to make another remark about the order of arguments. To this end, consider, for simplicity, a tensor T of type $(0, 2)$. Its image on the vectors $\mathbf{u}$ and $\mathbf{v} \in X$ is $T(\mathbf{u}, \mathbf{v})$. Now, define $S: X \times X \to \mathbb{R}$ by $S(\mathbf{u}, \mathbf{v}) \coloneqq T(\mathbf{v}, \mathbf{u})$ for all $\mathbf{u}, \mathbf{v} \in X$. Clearly, S is also a tensor of type $(0, 2)$. The components of T are

$$T_{ab} \stackrel{\text{def}}{=} T(\mathbf{e}_a, \mathbf{e}_b)$$

while the components of S are

$$S_{ab} \stackrel{\text{def}}{=} S(\mathbf{e}_a, \mathbf{e}_b) = T(\mathbf{e}_b, \mathbf{e}_a) \stackrel{\text{def}}{=} T_{ba}$$

where the $\stackrel{\text{def}}{=}$'s are the definition of the components of a tensor. The subscripts a and b in (↑) and (↑) are the same; for example, $T_{12} = S_{21}$.

5.1.4 Tensorial Operations

5.1.4.1 Tensor Addition

It is natural to make

Definition NN

Let A and B be two tensors of the same type (k, l). Then the *tensor sum* of A and B is the tensor $A + B$ of type (k, l) defined by

$$(A + B)(u_1, \dots, u_k, \mathbf{v}_1, \dots, \mathbf{v}_l) \coloneqq A(u_1, \dots, u_k, \mathbf{v}_1, \dots, \mathbf{v}_l) + B(u_1, \dots, u_k, \mathbf{v}_1, \dots, \mathbf{v}_l)$$

where the u_i's are arbitrary covectors and the $\mathbf{v}_i$'s are arbitrary vectors.

It is trivial to see that $A + B$ actually *is* a tensor, and that $A + B = B + A$. We want to express the components of $C \coloneqq A + B$ in terms of the components of A and B (given a basis, of course). This is easy, for

$$C_{ab\dots}^{ij\dots} \overset{\text{def}}{=} C(\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j, \dots, \mathbf{e}_a, \mathbf{e}_b, \dots) \overset{\text{def}}{=} A(\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j, \dots, \mathbf{e}_a, \mathbf{e}_b, \dots) + B(\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j, \dots, \mathbf{e}_a, \mathbf{e}_b, \dots) \overset{\text{def}}{=} A_{ab\dots}^{ij\dots} + B_{ab\dots}^{ij\dots}$$

where the first and the last $\overset{\text{def}}{=}$ is the definition of the component symbol, and the middle $\overset{\text{def}}{=}$ is the definition of the tensor sum.

5.1.4.2 Multiplication by a Scalar

It is also natural to make

Definition NN

Let A be a tensor of type (k, l), and let $\lambda \in \mathbb{R}$ be any number. Then λA is the tensor of type (k, l) defined by

$$(\lambda A)(u_1, \dots, u_k, \mathbf{v}_1, \dots, \mathbf{v}_l) \coloneqq \lambda A(u_1, \dots, u_k, \mathbf{v}_1, \dots, \mathbf{v}_l)$$

where the u_i's are arbitrary covectors and the $\mathbf{v}_i$'s are arbitrary vectors.

Almost needless to say, the components of $C \coloneqq \lambda A$ are those of A multiplied by λ, for

$$C_{ab\dots}^{ij\dots} \overset{\text{def}}{=} C(\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j, \dots, \mathbf{e}_a, \mathbf{e}_b, \dots) \overset{\text{def}}{=} \lambda A(\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j, \dots, \mathbf{e}_a, \mathbf{e}_b, \dots) \overset{\text{def}}{=} \lambda A_{ab\dots}^{ij\dots}.$$

We remark that we have now shown that the tensors of any specific type (k, l) form a vector space over the real numbers, under the operations of tensor addition and multiplication by a scalar. (Of course, one has to check the rest of the vector space axioms, but they are trivial.)

5.1.4.3 Tensor Multiplication

The definition of the tensor product is very simple.

Definition NN

Let A and B be two tensors of types (k_A, l_A) and (k_B, l_B), respectively. Then the *tensor product* of A and B is the tensor $A \otimes B$ of type $(k_A + k_B, l_A + l_B)$ defined by

$$\begin{aligned}(A \otimes B)(u_1, \dots, u_{k_A}, \dots, u_{k_A+k_B}, \mathbf{v}_1, \dots, \mathbf{v}_{l_A}, \dots, \mathbf{v}_{l_A+l_B}) \coloneqq \\ \coloneqq A(u_1, \dots, u_{k_A}, \mathbf{v}_1, \dots, \mathbf{v}_{l_A}) \cdot B(u_{k_A+1}, \dots, u_{k_A+k_L}, \mathbf{v}_{l_A+1}, \dots, \mathbf{v}_{l_A+l_B})\end{aligned}$$

where the u_i's are arbitrary covectors and the $\mathbf{v}_i$'s are arbitrary vectors.

Given a basis of X, we want to find the components of $C \coloneqq A \otimes B$ in terms of the components of A and B. This is very easy. Indeed,

$$C^{ij\cdots\mu\nu\cdots}_{ab\cdots\alpha\beta\cdots} \overset{\text{def}}{=} C(\boldsymbol{\sigma}^i, \ldots, \boldsymbol{\sigma}^\mu, \ldots, \mathbf{e}_a, \ldots, \mathbf{e}_\alpha, \ldots) \overset{\text{def}}{=} A(\boldsymbol{\sigma}^i, \ldots, \mathbf{e}_a, \ldots) \cdot B(\boldsymbol{\sigma}^\mu, \ldots, \mathbf{e}_\alpha, \ldots) \overset{\text{def}}{=} A^{ij\cdots}_{ab\ldots} B^{\mu\nu\cdots}_{\alpha\beta\cdots}$$

where the first and the last $\overset{\text{def}}{=}$ is the definition of the component symbol, and the middle $\overset{\text{def}}{=}$ is the definition of the tensor product. (We also pretend that there is no lack of letters.) That is, the components of $A \otimes B$ are the products of the components of A and B, in the sense above.

A tensor T of type (k, l) is said to be *simple* if there is a number $\lambda \in \mathbb{R}$, k vectors $\mathbf{u}$, $\mathbf{v}$, ..., and l covectors $f, g, \ldots$ such that

$$T = \lambda\, (\mathbf{u} \otimes \mathbf{v} \otimes \cdots \otimes f \otimes g \otimes \cdots).$$

Proposition NN

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a basis for X, and let $\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^n$ be the dual basis. Then the n^{k+l} simple tensors

$$\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_l} \otimes \boldsymbol{\sigma}^{j_1} \otimes \cdots \otimes \boldsymbol{\sigma}^{j_k}$$

where $i_p \in [1..n], \forall p \in [1..l]$ and $j_q \in [1..n], \forall q \in [1..k]$ form a basis for the vector space of (k, l)-tensors over X. In addition, the components of a tensor T are also the coordinates of T in this vector space and given this basis.

Proof

Let T be any tensor of type (k, l) and let

$$T^{i_1\cdots i_k}_{j_1\cdots j_l} \overset{\text{def}}{=} T(\boldsymbol{\sigma}^{i_1}, \ldots, \boldsymbol{\sigma}^{i_k}, \ldots, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_l})$$

be its components. Then the image[54]

$$T(f, \ldots, g, \mathbf{u}, \ldots, \mathbf{v}) = T^{i_1\cdots i_k}_{j_1\cdots j_l} f_{i_1} \ldots g_{i_k} u^{j_1} \ldots v^{j_l}$$

We want to find n^{k+l} unique scalars $\lambda^{i_1\cdots i_k}_{j_1\cdots j_l}$ such that

$$T = \lambda^{i_1\cdots i_k}_{j_1\cdots j_l} (\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k} \otimes \boldsymbol{\sigma}^{j_1} \otimes \cdots \otimes \boldsymbol{\sigma}^{j_l}),$$

that is, such that the image

$$T(f, \ldots, g, \mathbf{u}, \ldots, \mathbf{v}) = \lambda^{i_1\cdots i_k}_{j_1\cdots j_l} (\mathbf{e}_{i_1}(f) \cdots \mathbf{e}_{i_k}(g) \boldsymbol{\sigma}^{j_1}(\mathbf{u}) \cdots \boldsymbol{\sigma}^{j_l}(\mathbf{v}) \cdots) = \lambda^{i_1\cdots i_k}_{j_1\cdots j_l} (f_{i_1} \ldots g_{i_k} u^{j_1} \ldots v^{j_l})$$

for all covectors $f, g, \ldots$ and vectors $\mathbf{u}, \mathbf{v}, \ldots$. The only possibility then is that

$$\lambda^{i_1\cdots i_k}_{j_1\cdots j_l} = T^{i_1\cdots i_k}_{j_1\cdots j_l};$$

thus, every tensor T can be written as a linear combination of the proposed basis vectors in a unique way, and the statement follows. ■

5.1.4.4 Tensor Contraction

A tensorial operation that, at first sight, is less obvious is that of *contraction*, a *unary* tensorial operation. For one thing, contraction is only defined for mixed tensors.

[54] The author is perfectly aware of the fact that there are not infinitely many letters between 'f' and 'g' in the English alphabet, and not between 'u' and 'v' either. However, the notation is bloated as it is, and so it is not particularly tempting to make it even more bloated by writing the covectors $f_1, f_2, \ldots$ and their components $(f_1)_1, (f_1)_2, \ldots$ and so on.

Definition NN

Let T be any *mixed* tensor of type (k, l), and let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis for X, and let $\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n$ be the dual basis. The *contraction* of T with respect to the ith covector slot and the jth vector slot ($i \in [1..k], j \in [1..l]$) is the tensor

$$\sum_{p=1}^{n} T(u_1, \dots, \underbrace{\boldsymbol{\sigma}^p}_{i}, \dots, u_k, \mathbf{v}_1, \dots, \underbrace{\mathbf{e}_p}_{j}, \dots, \mathbf{v}_l)$$

of type $(k-1, l-1)$.

For example, if T is a tensor of type $(2, 3)$, then the contraction with respect to the second covector slot and the second vector slot is the (1,2)-tensor with image

$$T'(f, \mathbf{u}, \mathbf{v}) = \sum_{p=1}^{n} T\left(f, \boldsymbol{\sigma}^p, \mathbf{u}, \mathbf{e}_p, \mathbf{v}\right), \qquad f \in X^*, \qquad \mathbf{u}, \mathbf{v} \in X.$$

As usual, we are interested in the components of the resulting tensor. In the example above,

$$T'^{a}_{ce} \overset{(1)}{=} T'(\boldsymbol{\sigma}^a, \mathbf{e}_c, \mathbf{e}_e) \overset{(2)}{=} \sum_{p=1}^{n} T\left(\boldsymbol{\sigma}^a, \boldsymbol{\sigma}^p, \mathbf{e}_c, \mathbf{e}_p, \mathbf{e}_e\right) \overset{(1)}{=} \sum_{p=1}^{n} T^{ap}_{cpe} \overset{(3)}{=} T^{ap}_{cpe}$$

where (1) is the definition of the components of a tensor, (2) is the definition of the contraction T', and (3) is the Einstein convention. Thus, if we have a tensor with components

$$T^{ab}_{cde},$$

then the components of the contraction w.r.t. the indices b and d are

$$T^{ap}_{cpe}.$$

It should be clear that, in general, one obtains the components of a contracted tensor by starting with the components of the original tensor, and then setting the contracted indices equal (thereby transforming the indices into dummy indices of an implied summation). This results in a tensor with one less upper index and one less lower index.

Notice that, in the case of a linear transformation T with components T^a_b, the only possible contraction is the trace $T^a_a = \operatorname{tr} T$, a scalar. Thus, the operation of contraction generalises that of taking the trace of a linear transformation.

From Definition NN it looks like the contraction of a tensor depends upon the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ you use to compute the contraction. In other words, it looks like you get different tensors in different bases. This is not the case, however. You know this in the special case of the trace of a linear transformation, and we now prove it in the general case.

Proposition NN

The contraction of a tensor does not depend upon the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ used to construct it.

Proof

TBW.

5.1.5 The Abstract Index Notation

If T is a tensor of type (k, l), then we have defined

$$T^{ij\cdots}_{ab\cdots} \overset{\text{def}}{\equiv} T(\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j, \ldots, \mathbf{e}_a, \mathbf{e}_b, \ldots)$$

as a general component of T with respect to the chosen basis (or bas*es*, if you insist that the dual basis, which is determined by the basis of X, is a different basis). For example, if

$$\mathbf{v} = 5\mathbf{e}_1 + 7\mathbf{e}_2 + 9\mathbf{e}_3$$

then $v^1 = 5$, $v^2 = 7$, and $v^3 = 9$ in the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. However, it is common to denote the *tensor itself* by the formal symbol $T^{ij\cdots}_{ab\cdots}$, where the indices are *not* substituted, or meant to be substituted, by actual numbers. For example, we may write

$$v^a(u)$$

for the image of $u \in X^*$ under $v^a \in X$.

This convention is called the *abstract index notation*, since the indices are now only formal symbols, telling us what type of tensor we are working with, and giving names to the different arguments (or slots). There are several benefits of the abstract index notation. First, it reduces the 'load on the box of symbols', or the 'pollution of the namespace'. For example, if $\mathbf{v}$ is a vector and f is its induced covector [that is, $f(\mathbf{u}) \coloneqq \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u} \in X$], then we can denote the vector $\mathbf{v}$ by v^a and the covector f by v_a. Another, rather obvious, benefit of the abstract index notation is that you can see immediately what type a tensor is. Indeed, if T is a tensor, we have no indication of its type whatsoever, but if we write it as T^{ab}_c, then we see that it is of type $(2, 1)$ and of rank $2 + 1 = 3$. Yet another reason is that the notation gives us a chance to give names to the different slots of the tensor. For instance, we will use T_{ba} as the tensor T_{ab} with its arguments swapped.

From now on, we will employ the abstract index notation. However, when we actually *are* referring to concrete components relative to some *specific* basis, we will use *Greek* letters as indices; we reserve Latin letters for abstract indices. To make this very clear, I offer the following diagram:

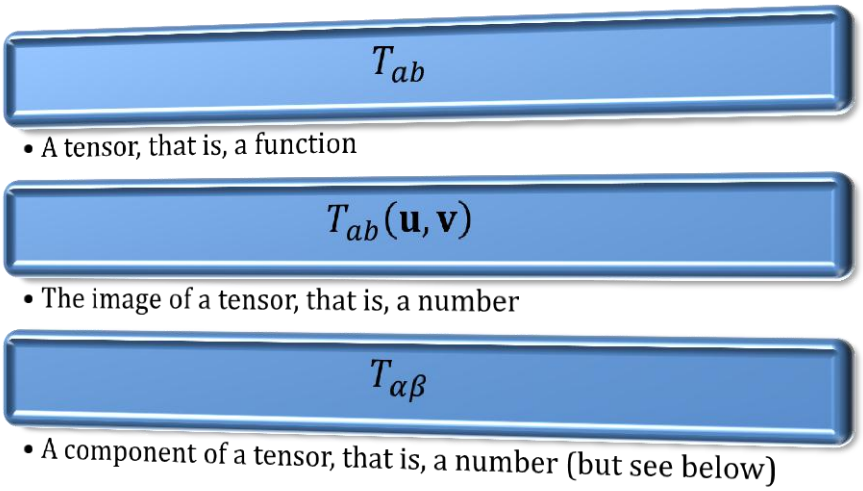


Figure 49. The Abstract Index Notation explained.

We have seen that, if $T = T_{ab}$ is a tensor with components $T_{\alpha\beta}$, then the tensor S obtained from T by swapping the order of the arguments has components $T_{\beta\alpha}$. In the abstract index notation, we can safely use the same letter for these two tensors. Indeed, we can certainly tell T_{ab} apart from T_{ba}. Formally, if there we introduce a tensor $T^{abc\cdots}_{ijk\cdots}$ and then start to talk about a tensor with two abstract indices swapped, $T^{acb\cdots}_{ijk\cdots}$ say, we mean the first tensor but with the corresponding arguments swapped.

A very nice feature of the abstract index notation is that it helps us not making mistakes. For example, it is obvious that we have made a mistake if we end up with $T^{ab} = S_{ab}$, since a contravariant tensor can never be the same thing as a covariant tensor; indeed, they live in different vector spaces. However, it is important to realise that it is perfectly valid to end up with a result like $T^{\alpha\beta} = S_{\alpha\beta}$, since each side is merely a number, and all numbers live in the same space (namely, $\mathbb{R}$)! Perhaps $T^{\alpha\beta} = 5$ and $S_{\alpha\beta} = 5$. Then, clearly, $T^{\alpha\beta} = S_{\alpha\beta}$.

In the special case of a tensor of rank one or two, it is possible to arrange its components (given a particular basis) in a one-dimensional or two-dimensional array of numbers (respectively). For pure convenience, we will use the convention of denoting such a matrix by the Greek index variant of the tensor symbol. For example,

$$v^\alpha = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

by definition, has exactly the same meaning as

$$v^1 = 1, \qquad v^2 = 0, \qquad v^3 = 1.$$

Notice that, in this case, the α is not meant to be replaced by an actual number, even though it is a Greek index! Instead, the Greek superscript indicates what kind of tensor we are talking about, and that the symbol represents *the components of the tensor in a particular basis*. We will *never* write

$$v^a = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{We will never write this!}$$

because v^a is a tensor (a contravariant vector, in this case), and we make a clear distinction between such a geometrical object and its components in any particular basis. Similarly, for a covector (and a prescribed dual basis), we may write

$$v_\alpha = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

which has exactly the same meaning as

$$v_1 = 1, \qquad v_2 = 0, \qquad v_3 = 0.$$

Finally, for any tensor of rank 2 [of type $(1,1)$, $(2,0)$, or $(0,2)$], we may write its components in matrix notation. For example,

$$T_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

has the exact same meaning as

$$T_{11} = 1, \quad T_{12} = 0, \quad T_{13} = 0, \quad T_{21} = 0, \quad T_{22} = 1, \quad T_{23} = 0, \quad T_{31} = 0, \quad T_{32} = 0, \quad T_{33} = -1.$$

Notice in particular that the symbol $u_\alpha v^\alpha$ now can be interpreted in two different ways, both of which are valid. Either we may consider

$$u_\alpha v^\alpha = \sum_{\alpha=1}^{n} u_\alpha v^\alpha = u_1 v^1 + \cdots + u_n v^n$$

where each u_α and v^α is a number, or we might think of it as

$$u_\alpha v^\alpha = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} = u_1 v^1 + \cdots + u_n v^n.$$

In either case, we end up with the same number, namely, $u_\alpha v^\alpha = u(v) \in \mathbb{R}$.

5.1.5.1 *The Basis Vectors* are *Vectors*

Be sure to notice that a single basis vector, like $\mathbf{e}_a$, *is* a vector; more precisely,

$$\mathbf{e}_a = 0\mathbf{e}_1 + \cdots + 1\mathbf{e}_a + \cdots + 0\mathbf{e}_n.$$

Hence, it has components

$$(\mathbf{e}_a)^\beta = [\beta = a]$$

and is written

$$(\mathbf{e}_a)^b$$

in the abstract index notation. The same remark holds for the dual space basis vectors.

5.1.6 *Symmetries of Tensors*

Let T_{ab} be a tensor of type $(0, 2)$, as indicated. It might so happen that

$$T_{ab}(\mathbf{u}, \mathbf{v}) = T_{ab}(\mathbf{u}, \mathbf{v}), \qquad \forall \mathbf{u}, \mathbf{v} \in X$$

in which case T_{ab} is said to be *symmetric* with respect to its indices a and b. It might also be the case that

$$T_{ab}(\mathbf{u}, \mathbf{v}) = -T_{ab}(\mathbf{u}, \mathbf{v}), \qquad \forall \mathbf{u}, \mathbf{v} \in X$$

in which case T_{ab} is said to be *antisymmetric* with respect to the same indices. In components,

$$T_{\alpha\beta} = T_{\beta\alpha}$$

and

$$T_{\alpha\beta} = -T_{\beta\alpha}$$

in each case, respectively. To see this, simply employ the definition of the component of a tensor:

$$T_{\alpha\beta} \overset{\text{def}}{=} T_{ab}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = T_{ab}(\mathbf{e}_\beta, \mathbf{e}_\alpha) \overset{\text{def}}{=} T_{\beta\alpha}$$

and

$$T_{\alpha\beta} \overset{\text{def}}{=} T_{ab}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = -T_{ab}(\mathbf{e}_\beta, \mathbf{e}_\alpha) \overset{\text{def}}{=} -T_{\beta\alpha}$$

in the two cases, respectively. The concepts of symmetric and antisymmetric indices are also defined for arbitrary tensors, not just tensors of type $(0, 2)$. The generalisation is rather obvious:

A tensor is symmetric (resp. antisymmetric) in two vector or covector slots if the image of the tensor is the same (resp. changes sign) if the arguments are swapped, for all possible arguments of the slots and for all values of the other arguments of the tensor.

5.1.6.1 The Symmetric and Antisymmetric Parts of a Tensor

Given a tensor T_{ab} of type $(0,2)$, we define two new tensors $T_{(ab)}$ and $T_{[ab]}$ of the same type by

$$\begin{aligned} T_{(ab)}(\mathbf{u},\mathbf{v}) &\coloneqq \frac{1}{2}\big(T_{ab}(\mathbf{u},\mathbf{v}) + T_{ab}(\mathbf{v},\mathbf{u})\big) = \\ &= \frac{1}{2}\big(T_{ab}(\mathbf{u},\mathbf{v}) + T_{ba}(\mathbf{u},\mathbf{v})\big) \quad \text{and} \\ T_{[ab]}(\mathbf{u},\mathbf{v}) &\coloneqq \frac{1}{2}\big(T_{ab}(\mathbf{u},\mathbf{v}) - T_{ab}(\mathbf{v},\mathbf{u})\big) = \\ &= \frac{1}{2}\big(T_{ab}(\mathbf{u},\mathbf{v}) - T_{ba}(\mathbf{u},\mathbf{v})\big); \end{aligned}$$

these tensors are the *symmetric* and the *antisymmetric part* of T_{ab}, respectively. The reason for this terminology is pretty obvious. Indeed,

$$T_{ab} = T_{(ab)} + T_{[ab]}$$

and so $T_{(ab)}$ and $T_{[ab]}$ are 'parts' of T_{ab}, and

$$T_{(ab)}(\mathbf{u},\mathbf{v}) = T_{(ab)}(\mathbf{v},\mathbf{u}), \qquad \forall \mathbf{u},\mathbf{v} \in X$$

while

$$T_{[ab]}(\mathbf{u},\mathbf{v}) = -T_{[ab]}(\mathbf{v},\mathbf{u}), \qquad \forall \mathbf{u},\mathbf{v} \in X;$$

that is, $T_{(ab)}$ is symmetric while $T_{[ab]}$ is antisymmetric. The components of the symmetric and antisymmetric parts look the way one might expect. The components of the symmetric part $T_{(a,b)}$ are

$$\frac{1}{2}\big(T_{\alpha\beta} + T_{\beta\alpha}\big) =: T_{(\alpha\beta)}$$

and the components of the antisymmetric part $T_{[a,b]}$ are

$$\frac{1}{2}\big(T_{\alpha\beta} - T_{\beta\alpha}\big) =: T_{[\alpha\beta]}.$$

Notice that if it so happens that T_{ab} *is* symmetric, then it is equal to its symmetric part:

$$T_{ab} = T_{(ab)}, \qquad T_{[ab]} = 0.$$

If it so happens, instead, that T_{ab} is antisymmetric, then

$$T_{ab} = T_{[ab]}, \qquad T_{(ab)} = 0.$$

Naturally, we define the symmetric and antisymmetric parts of a contravariant tensor T^{ab} analogously. Furthermore, in a tensor of higher rank than two, if a *neighbouring pair* of indices on the same vertical level are enclosed by parenthesis (or square brackets), we mean that we should compute the value of the tensor twice, using both orders of the two arguments corresponding to the indices inside parenthesis or brackets, and then sum (resp. subtract) half the obtained values. For example,

$$T_{de}^{a(bc)} = \frac{1}{2}\big(T_{de}^{abc} + T_{de}^{acb}\big).$$

More generally, if there are n arguments inside parentheses, we sum over every possible permutation of the corresponding arguments. If the indices are in square brackets instead of parentheses, we do the same, but we also multiply every term by the sign of the corresponding permutation. In any case, we divide the result by $n!$, the number of permutations. That is,

$$T_{(a_1,a_2,\ldots,a_n)} \coloneqq \frac{1}{n!}\sum_{\sigma\in S_n} T_{a_{\sigma(1)},a_{\sigma(2)},\ldots,a_{\sigma(n)}}$$

where S_n is the symmetric group on $\{1, 2, \ldots, n\}$, and

$$T_{[a_1,a_2,\ldots,a_n]} \coloneqq \frac{1}{n!}\sum_{\sigma\in S_n} \operatorname{sgn}\sigma \cdot T_{a_{\sigma(1)},a_{\sigma(2)},\ldots,a_{\sigma(n)}}$$

where $\operatorname{sgn}\colon S_n \to \{-1, +1\}$ is defined by

$$\operatorname{sgn}\sigma = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}, \qquad \forall\sigma \in S_n.$$

We make the same definition for a contravariant tensor of rank n, and we also make the obvious generalisation to an arbitrary tensor: If, in a tensor of rank k, there are n of the indices inside parenthesis or square brackets ($n \leq k$), we sum over all permutations of the corresponding arguments, leaving the other arguments fixed.

In some cases, not even this very general notation is enough. Say, for instance, that we have a tensor T_{abc} and are interested in

$$\frac{1}{2}(T_{abc} + T_{cba}).$$

In this case, we want to swap the two indices a and c while keeping b fixed, but unfortunately, a and c are on opposite sides of b. To remedy this, we introduce a new device, and write

$$T_{(a\hat{b}c)} = \frac{1}{2}(T_{abc} + T_{cba})$$

where the hat (or 'home') ^ on b tells us that b is 'at home', and not going to move anywhere. Of course, this device is used not only for symmetric parts of covariant tensors of rank 3, but on all tensors and both together with parentheses (symmetric parts) and square brackets (antisymmetric parts).

5.1.7 Transformation Properties of Tensor Components

If $\underline{\mathbf{e}} = (\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n)$ and $\underline{\mathbf{f}} = (\mathbf{f}_1 \quad \cdots \quad \mathbf{f}_n)$ are two bases in X, then a tensor T on X will have different components in the two bases, just as a vector, or a linear transformation, has different components. We will review the terminology from elementary linear algebra before we approach the transformation of a general tensor using tensor notation.

If we change from $\underline{\mathbf{e}}$ to $\underline{\mathbf{f}}$, all information about the change of basis is encoded into the $n \times n$ 'change-of-basis' matrix C (with elements C_i^j) defined by the formal matrix equality[55]

[55] When we specify the elements of a matrix using index notation, then, the following rules apply: If the indices are on the same height (either superscript or subscript), then the left-most index is the row index and the right-most index is the column index. If not, the upper index is the row index while the lower index is the column index. For example, in each of the following symbols, i is the row index and j is the column index of the matrix: A^{ij}, A_{ij}, A^1_{ij}, A^i_j, A^{2i}_j. (Beware that some authors employ a different convention.)

$$\underline{\mathbf{f}} = \underline{\mathbf{e}}C = \underline{\mathbf{e}}\begin{pmatrix} C_1^1 & \cdots & C_n^1 \\ \vdots & \ddots & \vdots \\ C_1^n & \cdots & C_n^n \end{pmatrix}.$$

More rigorously put, the jth column of C consists of the coordinates of $\mathbf{f}_j$ expressed in the basis $\underline{\mathbf{e}}$. From elementary linear algebra, it is known that if a vector has components X (a column matrix) and X' in the old and the new basis, respectively, then

$$X' = C^{-1}X.$$

In general tensor analysis, we are forced to abandon the matrix notation, and the reason is quite simple: It is no longer enough with one-dimensional arrays of numbers ('vectors') and two-dimensional arrays of numbers ('matrices'); we need k-dimensional arrays of numbers, and these are difficult to represent. Thus, for instance, instead of representing a set of n^2 numbers as a matrix, we write out the numbers as components of a tensor by writing $T^{11} = \cdots$, $T^{12} = \cdots$, etc. This is readily generalised to k-dimensional arrays of numbers, for example, $T^{111} = \cdots$, $T^{112} = \cdots$, etc. We will now rewrite the transformation laws of vectors and linear transformations in such notation, and we will also derive the transformation law of a general tensor of any type.

5.1.7.1 The Transformation of a Vector

Let us now switch to tensor notation. The transformation law (↑) of a vector x^a can clearly be written

$$x'^{\alpha} = (C^{-1})^{\alpha}_{\beta}x^{\beta}.$$

5.1.7.2 The Transformation of a Covector

Consider now a covector x_a with coordinates x_α and x'_α in the old and the new bases, respectively. The transformation law of a covector can be deduced from that of a vector. Indeed, for *any* vector v^a, with components v^α and v'^α, respectively, the image

$$x_\alpha v^\alpha = x'_\alpha v'^\alpha.$$

But

$$v'^\alpha = (C^{-1})^\alpha_\beta v^\beta$$

and so

$$x_\alpha v^\alpha = x'_\alpha (C^{-1})^\alpha_\beta v^\beta$$

should hold for *any* vector v^a. If we rename the dummy (summation) indices,

$$x_\alpha v^\alpha = x'_\beta (C^{-1})^\beta_\alpha v^\alpha$$

and it is obvious that

$$x_\alpha = x'_\beta (C^{-1})^\beta_\alpha.$$

But this is tensor notation for a usual matrix multiplication; indeed, the equation says

$$(x_1 \quad \cdots \quad x_n) = (x'_1 \quad \cdots \quad x'_n)\begin{pmatrix} (C^{-1})^1_1 & \cdots & (C^{-1})^1_n \\ \vdots & \ddots & \vdots \\ (C^{-1})^n_1 & \cdots & (C^{-1})^n_n \end{pmatrix}.$$

But then, since C^{-1} is invertible,

$$(x_1 \quad \cdots \quad x_n)\begin{pmatrix} C_1^1 & \cdots & C_n^1 \\ \vdots & \ddots & \vdots \\ C_1^n & \cdots & C_n^n \end{pmatrix} = (x_1' \quad \cdots \quad x_n')$$

or, reverting to tensor notation,

$$C_\alpha^\beta x_\beta = x'_\alpha$$

or

$$\boxed{x'_\alpha = C_\alpha^\beta x_\beta}$$

which is the sought transformation law of a covector.

5.1.7.3 The Origin of the Words 'Covariant' and 'Contravariant'

We will make a brief recess in our endeavour of deriving the transformation properties of a general tensor and consider the etymologies of the words 'covariant' and 'contravariant'. So far, we have found the transformation of both a vector and a covector. Let us find the 'transformation law'[56] of a basis vector. This is actually (↑), which we can write, in tensor notation, as

$$\mathbf{f}_\alpha = \mathbf{e}_\beta C_\alpha^\beta,$$

or, if we let $\mathbf{f}_b =: \mathbf{e}'_b$,

$$\mathbf{e}'_\alpha = C_\alpha^\beta \mathbf{e}_\beta.$$

That is, the formal row matrix $(\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n)$ of basis vectors transform like the components of a single covector! (Be sure to see the difference.) Thus, a covector, or a covariant tensor of rank 1, transforms in the 'same way' as the basis, and 'co-' means, roughly, 'together'. On the contrary, a vector, or a contravariant tensor of rank 1 transforms in the opposite way, and 'contra-' means 'against'. This also motivates why we write the basis vectors like $\mathbf{e}_a$ with a downstairs index, just like we write covectors.

In addition, formal array of dual basis vectors transforms like the components of a single vector, which motivates our writing their indices upstairs.

5.1.7.4 The Transformation of a Linear Transformation

The classical transformation law (↑) of a linear transformation A_b^a can be written

$$\boxed{A'^\alpha_\beta = (C^{-1})^\alpha_\gamma C^\delta_\beta A^\gamma_\delta}$$

which therefore is the transformation law of a mixed tensor of type $(1,1)$. Strictly speaking, the above shows how the components of a linear transformation transforms, but recall that we in Section 5.1.3.4 showed that the components of a linear transformation are *identical* to those of the induced tensor of type (1,1), and so our statement follows.

5.1.7.5 The Transformation of a General Tensor

Let $T^{abc\cdots}_{ijk\cdots}$ be a general tensor with components

[56] The inverted commas are due to the fact that this 'law' is in fact a definition. Indeed, we defined our transformation from $\underline{\mathbf{e}}$ to $\underline{\mathbf{f}}$ by specifying the components of the new basis vectors in terms of the old ones.

$$T^{\alpha\beta\gamma\cdots}_{\mu\nu\xi\cdots} \overset{\text{def}}{=} T\left(\boldsymbol{\sigma}^\alpha, \boldsymbol{\sigma}^\beta, \boldsymbol{\sigma}^\gamma, \dots, \mathbf{e}_\mu, \mathbf{e}_\nu, \mathbf{e}_\xi, \dots\right)$$

relative to $\underline{\mathbf{e}}$ and

$$T'^{\alpha\beta\gamma\cdots}_{\mu\nu\xi\cdots} \overset{\text{def}}{=} T\left(\boldsymbol{\tau}^\alpha, \boldsymbol{\tau}^\beta, \boldsymbol{\tau}^\gamma, \dots, \mathbf{f}_\mu, \mathbf{f}_\nu, \mathbf{f}_\xi, \dots\right)$$

relative to $\underline{\mathbf{f}}$, where $\boldsymbol{\tau}^1, \dots, \boldsymbol{\tau}^n$ is the corresponding dual space basis. Using the known transformations

$$\boldsymbol{\tau}^\alpha = (C^{-1})^\alpha_\psi \boldsymbol{\sigma}^\psi, \qquad \mathbf{f}_\alpha = C^\psi_\alpha \mathbf{e}_\psi,$$

we have, by linearity,

$$\begin{aligned}
T'^{\alpha\beta\gamma\cdots}_{\mu\nu\xi\cdots} &\overset{\text{def}}{=} T\left(\boldsymbol{\tau}^\alpha, \boldsymbol{\tau}^\beta, \boldsymbol{\tau}^\gamma, \dots, \mathbf{f}_\mu, \mathbf{f}_\nu, \mathbf{f}_\xi, \dots\right) = \\
&= T\left((C^{-1})^\alpha_\psi \boldsymbol{\sigma}^\psi, (C^{-1})^\beta_\zeta \boldsymbol{\sigma}^\zeta, (C^{-1})^\gamma_\theta \boldsymbol{\sigma}^\theta, \dots, C^\kappa_\mu \mathbf{e}_\kappa, C^\lambda_\nu \mathbf{e}_\lambda, C^\rho_\xi \mathbf{e}_\rho, \dots\right) = \\
&= (C^{-1})^\alpha_\psi T\left(\boldsymbol{\sigma}^\psi, (C^{-1})^\beta_\zeta \boldsymbol{\sigma}^\zeta, (C^{-1})^\gamma_\theta \boldsymbol{\sigma}^\theta, \dots, C^\kappa_\mu \mathbf{e}_\kappa, C^\lambda_\nu \mathbf{e}_\lambda, C^\rho_\xi \mathbf{e}_\rho, \dots\right) = \cdots = \\
&= (C^{-1})^\alpha_\psi (C^{-1})^\beta_\zeta (C^{-1})^\gamma_\theta \cdots C^\kappa_\mu C^\lambda_\nu C^\rho_\xi \cdots T\left(\boldsymbol{\sigma}^\psi, \boldsymbol{\sigma}^\zeta, \boldsymbol{\sigma}^\theta, \dots, \mathbf{e}_\kappa, \mathbf{e}_\lambda, \mathbf{e}_\rho, \dots\right) = \\
&= (C^{-1})^\alpha_\psi (C^{-1})^\beta_\zeta (C^{-1})^\gamma_\theta \cdots C^\kappa_\mu C^\lambda_\nu C^\rho_\xi \cdots T^{\psi\zeta\theta\cdots}_{\kappa\lambda\rho\cdots}.
\end{aligned}$$

That is,

$$T'^{\alpha\beta\gamma\cdots}_{\mu\nu\xi\cdots} = (C^{-1})^\alpha_\psi (C^{-1})^\beta_\zeta (C^{-1})^\gamma_\theta \cdots C^\kappa_\mu C^\lambda_\nu C^\rho_\xi \cdots T^{\psi\zeta\theta\cdots}_{\kappa\lambda\rho\cdots}$$

is the transformation law of the components of a general tensor. It is easily checked that the laws for a vector, covector, and a linear transformation are special cases of the general law.

5.1.7.6 *The Classical Definition of a Tensor (Almost)*

Many older books, and many contemporary physics texts, use the 'coordinate approach' to tensors: they define a tensor, not as a real-valued multilinear function on a vector space, but simply as an array of numbers that transforms according to the laws above under a change of basis (almost). That is, they do (almost)

Alternative Definition NN

A *contravariant tensor of rank* k is a set of n^k numbers $a^{\alpha_1,\dots,\alpha_k}$ $(\alpha_i \in [1..n])$ such that, under a change of basis described by the matrix C, they transform as

$$(a')^{\alpha_1,\dots,\alpha_k} = (C^{-1})^{\alpha_1}_{\beta_1} \cdots (C^{-1})^{\alpha_k}_{\beta_k} a^{\beta_1,\dots,\beta_n}.$$

A *covariant tensor of rank* k is a set of n^k numbers $a_{\alpha_1,\dots,\alpha_k}$ $(\alpha_i \in [1..n])$ such that, under a change of basis, they transform as

$$a'_{\alpha_1,\dots,\alpha_k} = C^{\beta_1}_{\alpha_1} \cdots C^{\beta_k}_{\alpha_k} a_{\beta_1,\dots,\beta_2}.$$

A tensor of type (k, l) and rank $k + l$ is a set of n^{k+l} numbers $a^{\alpha_1,\dots,\alpha_k}_{\beta_1,\dots,\beta_k}$ $(\alpha_i, \beta_i \in [1..n])$ such that, under a change of basis, they transform as

$$(a')^{\alpha_1,\dots,\alpha_k}_{\beta_1,\dots,\beta_k} = (C^{-1})^{\alpha_1}_{\gamma_1} \cdots (C^{-1})^{\alpha_k}_{\gamma_k} C^{\delta_1}_{\beta_1} \cdots C^{\delta_k}_{\beta_k} a^{\gamma_1,\dots,\gamma_k}_{\delta_1,\dots,\delta_k}$$

where $k, l \in \mathbb{N}$ and the tensor is said to be *mixed* if $k, l > 0$.

An ordinary vector is often called a *contravariant vector*, while a covector is called a *covariant vector*. (We will return to the meaning of 'almost' later on, in Section 5.3.1.6.) One then generally shows that a curve tangent vector is a contravariant tensor of rank 1, and the reader might feel that the contravariant tensors are the most important kind of tensors, while the covariant tensors are some kind of 'technical pathology'. Indeed, everyone is familiar with vectors, but perhaps not as familiar with covectors. From the modern viewpoint, however, it is perfectly clear that in many cases the covariant tensors are in fact the most important ones. For example, the first fundamental form is a covariant tensor. Indeed, irrespective of the approach chosen when you define the concept of a tensor, tensors are generally *used* as linear functionals. For example, even if you only consider tensors as arrays of numbers, a covariant tensor with components $T_{\alpha\beta}$ generally 'acts' on two vectors with components v^α and u^β to yield a number $T_{\alpha\beta}v^\alpha u^\beta$. That is, even if you don't use that exact phrasing, you are using the numbers as a linear functional! Now, if you consider vectors to be of more 'important' than covectors, it is 'clear' that covariant tensors are more 'important' than contravariant tensors, because only the former accepts vectors as arguments.

For future use, we end this section by giving

Definition NN

The *Kronecker tensor* is the tensor δ_b^a with image

$$\delta(u, \mathbf{v}) = u(\mathbf{v}) = u_a v^a, \qquad \forall u \in X^*, \mathbf{v} \in X.$$

We give some of the most important properties of δ_b^a.

Proposition NN

(1) δ_b^a is the tensor induced by the identity linear transformation.

(2) In *any* basis, $\delta_\beta^\alpha = [\alpha = \beta]$, which is known as the *Kronecker delta symbol*.

(3) $\delta_\beta^\alpha \delta_\gamma^\beta = \delta_\gamma^\alpha$.

(4) $(\mathbf{e}_\alpha)^\beta = \delta_\alpha^\beta$.

Proof

(1): If A is the identity linear transformation, then the tensor induced from A has image $u(A\mathbf{v}) = u(\mathbf{v})$ since A is the identity matrix. (2): $\delta_\beta^\alpha \overset{\text{def}}{=} \delta(\boldsymbol{\sigma}^\alpha, \mathbf{e}_\beta) = \boldsymbol{\sigma}^\alpha(\mathbf{e}_\beta) = [\alpha = \beta]$. (3) is immediate and (4) is (↑) written in disguise using (2). ■

5.1.8 The Metric

A general vector space has no inner product, but in this section we will see that if an inner product has been prescribed, much more structure can be introduced when we consider tensors on the space. As shown in Section 5.1.1.2, an inner product in a vector space X can always be written

$$\langle u, v\rangle = g_{ab}u^a v^b$$

for some non-degenerate, symmetric, and positive definite matrix g_{ab}. Clearly, the inner product $\langle\cdot,\cdot\rangle\colon X\cdot X\to\mathbb{R}$ is a covariant tensor of rank 2, and $g_{\mu\nu}$ are its components. Indeed,

$$\langle \mathbf{e}_\mu, \mathbf{e}_\nu\rangle = g_{\alpha\beta}(\mathbf{e}_\mu)^\alpha(\mathbf{e}_\nu)^\beta = g_{\alpha_\beta}\delta^\alpha_\mu\delta^\beta_\nu = g_{\mu\nu}.$$

This tensor, generally denoted g_{ab}, is called the *metric tensor*, or the *metric* for short[57]. The inverse of g_{ab}, considered a matrix (which exists since g_{ab} is non-degenerate), is denoted g^{ab}. The coordinate-free definition of g^{ab} is given by

$$g_{ab}g^{bc} = \delta^c_a.$$

We have already observed that g_{ab} yields a natural correspondence between vectors and covectors: Any vector v^a yields a unique covector $f\colon X\to\mathbb{R}$ defined by

$$f(w) \coloneqq \langle w, v\rangle = g_{ab}v^a w^b, \qquad \forall w\in X,$$

which is clearly written $f_b = g_{ab}v^a$ in the index notation (why?). *We will write this simply as*

$$v_b \coloneqq g_{ab}v^a$$

and we may say that "v_b is the covector induced by the contravariant vector v^a". We also say that we have used the metric g_{ab} to 'lower the index' of v^a. Similarly, given a covector u_a, the inner product yields a unique contravariant vector v^a by requiring that

$$u(w) = \langle w, v\rangle, \qquad \forall w\in X.$$

In index notation

$$u_a w^a = g_{ab}v^b w^a;$$

thus,

$$u_a = g_{ab}v^b.$$

Notice that u_a is what you get is you lower the index of v^b. It follows[58] that

$$v^c = g^{ac}u_a.$$

and we write

$$u^c \coloneqq g^{ac}u_a.$$

We say that "u^c is the contravariant vector induced by the covector u_a", and that we have used the inverse metric g^{ac} to 'raise the index' of u_a. If this procedure is to be self-consistent, it is necessary, for instance, that you get a vector back if you lower its index and then raise it again. But that is what we noticed in (↑). The procedure of 'raising and lowering indices' generalises to any tensors. For example,

$$T^{ab}{}_c{}^f \coloneqq g^{df}T^{ab}{}_{cd}.$$

Again, if our notation is to be self-consistent, it is necessary that g^{ab}, as defined by (↑), actually *is* the unique tensor of type $(2,0)$ that you obtain from g_{ab} by raising both indices using g^{ab}. Fortunately, this is so, because

[57] However, the 'metric tensor' is *not* a metric on X in the usual sense of algebra and functional analysis; instead, it is an inner product.
[58] Formally, $u_a = g_{ab}v^b \Rightarrow g^{ac}u_a = g^{ac}g_{ab}v^b = \delta^c_b v^b = v^c$.

$$g^{bc}g_{ab} = \delta^c_a \Rightarrow g^{ad}g^{bc}g_{ab} = g^{ad}\delta^c_a = g^{cd}.$$

It is clear that any tensor of rank k belongs to an equivalence class consisting of 2^k tensors that differs only in the vertical positions of their indices. For example, a tensor T_{ab} belongs to the same class as T^{ab}, $T_a{}^b$, and $T^a{}_b$. If T and S are two tensors, we write $T \sim S$ iff they belong to the same equivalence class, that is, if the tensors are equal if their indices are placed on the same heights. For technical reasons, we are particularly interested in the tensor δ^a_b. We have

Proposition NN

$$g \sim \delta;$$

in particular,

$$\begin{aligned} \delta_{ab} &= g_{ab}, \\ \delta^{ab} &= g^{ab}, \quad \text{and} \\ g^a{}_b &= \delta^a{}_b. \end{aligned}$$

In addition,

$$g^a_a = \delta^a_a = n \stackrel{\text{def}}{=} \dim X.$$

Proof

Equation (↑) reads

$$g^a{}_b = \delta^a{}_b$$

and

$$\delta^a_a = \sum_{i=1}^{n} \delta^a_b(\boldsymbol{\sigma}^i, \mathbf{e}_i) = \sum_{i=1}^{n} \boldsymbol{\sigma}^i(\mathbf{e}_i) = \sum_{i=1}^{n} 1 = n,$$

or, in terms of components,

$$\delta^\alpha_\alpha = \delta^1_1 + \cdots + \delta^n_n = 1 + \cdots + 1 = n.$$

■

5.1.8.1 Euclidean Space

In essentially all elementary mathematics (in particular, in elementary linear algebra) and physics, we use an orthonormal basis in $\mathbb{R}^n$. By definition, this means that $\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle = [\alpha = \beta]$. Thus,[59]

$$g_{\alpha\beta} \stackrel{\text{def}}{=} \left\langle (\mathbf{e}_\alpha)^\mu, (\mathbf{e}_\beta)^\nu \right\rangle = \delta^\alpha{}_\beta = [\alpha = \beta].$$

In this case there is no distinction at all between vectors and covectors, for

$$u_\alpha = g_{\alpha\beta}u^\beta = \delta^\alpha{}_\beta u^\beta = u^\alpha, \qquad \forall \alpha \in [1..n].$$

5.1.8.2 The Inner Product

Let u^a and v^b be two vectors. Then their inner product is

[59] If you think that this looks wrong, you forget that we are using Greek indices, not Latin (cf. the final paragraph of Section 5.1.5).

$$\langle u, v\rangle = g_{ab}u^a v^b = u_b v^b = u^a v_a$$

by raising or lowering the suitable indices. This is a special case of the following simple result, which actually is easier to prove than to formulate.

Proposition NN

In a tensorial term with a pair of equal indices, once upstairs and once downstairs [thus taking part in an Einstein summation], the tensor is unchanged if the vertical positions of each of the two indices are swapped.

Proof

$$u^a v_a = g^{ba}u_b v_a = u_b v^b = u_a v^a.$$

■

5.2 Manifolds

In this section, we will introduce the concept of a *manifold*, which is a generalisation of the notions of a curve (a one-dimensional manifold), a surface (a two-dimensional manifold), etc. to an arbitrary dimension. What is really 'new' is that, although (classically) a curve and a surface are always subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$, a general manifold need not to be given as a subset of some higher-dimensional space. In other words, in general, there is no 'outside' of the manifold.

A (topological) manifold is defined as a topological space (in particular, it is a *set*) with a number of postulated properties. Intuitively, the main property is that about every point in the manifold, it 'looks', locally, like $\mathbb{R}^n$, where n is independent of the point and is called the *dimension* of the manifold. For example, if you scrutinize a tiny region of a sphere, it looks very much like the plane $\mathbb{R}^2$. (Indeed, one used to believe the Earth was flat.) We will not indulge ourselves in all the technical details, so we will settle with

Definition NN

An n-dimensional *topological manifold* $M = M^n$ is a 'nice' topological space that is locally Euclidean, that is, every $x \in M$ has a neighbourhood homeomorphic to some open subset of $\mathbb{R}^n$.

If M is a manifold of dimension n, we will occasionally write it $M = M^n$ to indicate the dimension explicitly. If we introduce a manifold M^n and then speak of 'M' alone, we mean the very same manifold. By 'nice' we mean that it is not 'pathological'; for instance, $\mathbb{R}^n$ is certainly 'nice'. In more technical terms, 'nice' most often means 'Hausdorff' and either one of 'second-countable' or 'paracompact'. Since we will not be particularly interested in 'pathological' examples, we will simply say that a topological manifold is 'nice enough'. It is clear that a topological manifold generalises the notions of a manifold curve and a manifold surface, as given by Definitions NN and NN.

If $U \subseteq M$ is a sufficiently small open subset of M about any point $x \in M$, there exists an open subset $V \subseteq \mathbb{R}^n$ such that there exists a homeomorphism $\phi_U : U \to V$. The pair (U, ϕ_U) is called a *coordinate patch* (or *chart*), since U is a 'patch' of the manifold, and the function ϕ_U introduces a coordinate system on U. For example, you can consider U to be a small part of the Earth (not containing any of the poles), and ϕ_U to be a function that assigns the latitude and longitude to each point in U. Notice that, in the case of a surface $\Sigma \subset \mathbb{R}^3$ covered by a single patch (U, ϕ_U), $\phi_U : M \to \mathbb{R}^2$ might be the *inverse* of the f-surface $\mathbf{r} : \mathbb{R}^2 \to M$.

We remark in particular that every point in a manifold has a neighbourhood homeomorphic to an *open* set in Euclidean space. The requirement that the set be open implies that, although the open disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ certainly *is* a manifold, the closed disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is *not* (it is a so-called 'manifold with boundary'). This requirement simplifies matters, since we do not need to treat 'boundary points' separately.

If (U, ϕ_U) and (V, ϕ_V) are two coordinate patches such that $U \cap V \neq \emptyset$, we also require that $U \cap V$ is open and that the *transition function* $f_{VU} \coloneqq \phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \to \phi_V(U \cap V)$ is a diffeomorphism, as depicted in Figure 50. [Thus, $f_{UV}(x)$ is the (V, ϕ_V) coordinate of the point in the manifold which has x as its (U, ϕ_U) coordinate.] In such case, we say that the patches are *compatible*. A collection $\left(U_i, \phi_{U_i}\right)$ of compatible charts such that $\bigcup_i U_i = M$ is called an *atlas* for M. A maximal collection of such charts is called a maximal atlas, and is denoted $\mathcal{A}_M$. We also write $\mathcal{A}_{M,x} \coloneqq \{(U, \phi_U) \in \mathcal{A}_M : x \in U\}$ for a maximal set of such charts 'containing' $x \in M$. In practice,

we specify some initial coordinate patches when we define a particular manifold, and then the maximal atlas consisting of all charts compatible with these constitute *the* maximal atlas $\mathcal{A}_M$. We will therefore always assume that a given manifold has a unique maximal atlas.

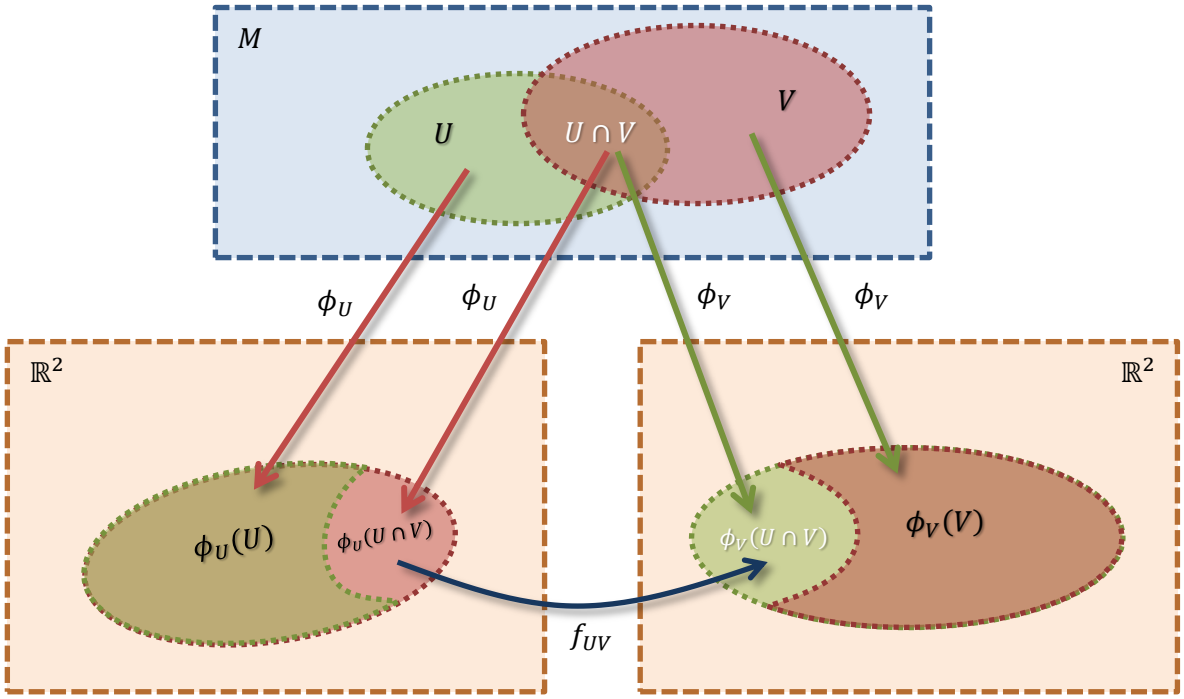


Figure 50: The transition function f_{UV}.

Definition NN

An n-dimensional topological manifold is called an n-dimensional *differential manifold* if it has an atlas of compatible coordinate charts.

Generally, we will assume 'diffeomorphic' to mean 'smooth diffeomorphic'. From now on, a 'manifold' will always be assumed 'differential'.

We also remark that the requirement that a manifold 'looks' like Euclidean space *locally* does not mean that it does so *globally*. Consider, for example, the cylinder $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, |z| < 1\}$ which is a differentiable manifold, and certainly looks like the Euclidean plane $\mathbb{R}^2$ at any point. Still, the area[60] of the cylinder is finite, while the area of $\mathbb{R}^2$ is infinite. In addition, there are closed curves on the cylinder that *cannot be continuously deformed to a single point*, which is highly 'odd' if you are only used to $\mathbb{R}^2$. The point is that the *topology* of an n-dimensional manifold may not resemble that of $\mathbb{R}^n$.

Notice that we can work with general n-dimensional manifolds in much the same way that we work with surfaces in space. Indeed, when we speak of a point on such a surface, we give its coordinates, and every point on a general manifold belongs to a coordinate patch that has a coordinate system on it. In particular, when we specify a point in a manifold, we generally do so by instead specifying the coordinates of the point in some local coordinate chart (U, ϕ_U). And when we specify a curve on a manifold M^n, instead of specifying a function $\gamma : I \to M$, we specify

[60] Here we allow ourselves to use the 'induced' metric properties of the ambient $\mathbb{R}^3$, which, of course, cannot be done in general.

$\phi_U \circ \gamma: I \to \mathbb{R}^n$. This, I think, is a very important point to realise; let us call it 'the coordinate patch observation', in lack of a better term.

We remark that our definition of a manifold does not admit self-intersecting curves and surfaces. Some of the 'pathologies' associated with such objects were discussed in Section 4.2.1. Now we can give yet another reason why we do not admit self-intersections: In physics, space (and, in relativity, spacetime), is modelled as a manifold of dimension three (resp. four). Consider now a curve intersecting itself, as in Figure 51.

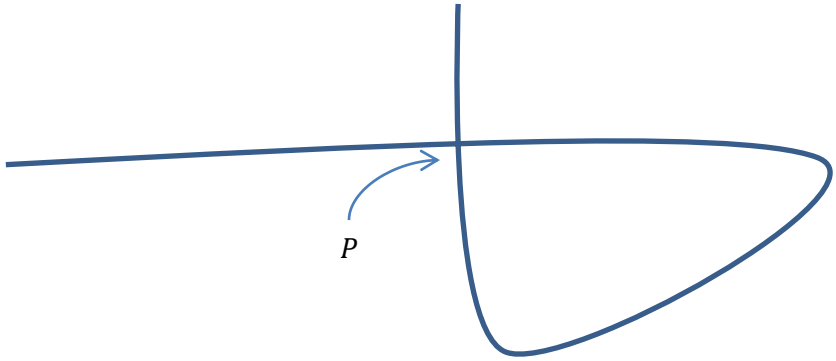


Figure 51. A curve intersecting itself.

At almost every point on the curve, a bug can go walk in either of two directions: either 'forwards or 'backwards', that is, there is *one* linearly independent direction along which she can walk. However, at the particular *point P*, she can also choose to walk 'upwards' or 'downwards'. Similarly, new 'directions' appear along a *curve* of self-intersection on a self-intersecting surface. Now, imagine how surprised you would become if you were on your way to the bus and, suddenly, you find a *surface* in space along which there are *four* (linearly-independent) spatial directions along which you can move!

In a general manifold, of course, there is no notion of 'distance', 'angle', 'area', or 'volume'. The reason why we are able to speak about such concepts on curves and surfaces embedded in $\mathbb{R}^n$ is that $\mathbb{R}^n$ is an inner-product space, and the inner product induces a norm, and, in particular, a metric. This metric, in turn, induces a metric 'on' the curve or surface.

5.2.1 Functions on Manifolds

If M^n and N^m are two smooth manifolds, a function $f: M \to N$ is said to be *continuous* (*differentiable*, *smooth*, …), iff, at every point $x \in M$, there is a coordinate patch $(U, \phi_U) \in \mathcal{A}_{M,x}$ and a coordinate patch $(V, \phi_V) \in \mathcal{A}_{N,f(x)}$ such that the map $\phi_V \circ f \circ \phi_U^{-1}: \mathbb{R}^n \to \mathbb{R}^m$ is continuous (differentiable, smooth, …). Furthermore, if there are more than one pair of such coordinate patches, then the map $\phi_V \circ f \circ \phi_U^{-1}$ will be continuous (differentiable, smooth, …) for *every* possible pair of maps in each atlas, because of the compatibility requirement. This definition is very natural, almost obvious, if have learned to appreciate the 'coordinate patch observation'. Indeed, when we specify a function $f: M \to N$, we most frequently do this by actually specifying the map $\phi_V \circ f \circ \phi_U^{-1}$, perhaps even without thinking of it. For example, if the temperature on the Earth, which we consider a sphere of radius R, is

$$T(\theta) = a + b \sin\theta\,, \qquad a, b \in \mathbb{R}$$

at spherical coordinate $\theta \in [0, \pi]$, then we have specified a map not from the Earth itself, but from the parameter plane of the Earth, to the real numbers (which is a manifold itself).

A *curve* on a manifold M is a continuous function $f: I \to M$, where $I \subseteq \mathbb{R}$ is an *open* interval. Since such an interval is a manifold, the term 'continuous' is well-defined by the last paragraph. We might also refer to the image $f(I)$ as a curve; we will relax the usage of the 'f-' prefix somewhat, if there is no risk of confusion. Most often we are interested in differentiable and smooth curves. In fact, we will generally assume all functions to be as differentiable as required. As usual, we generally describe a curve by actually specifying $\phi_U \circ f: I \to \mathbb{R}^n$; for example, a part of the equator of the Earth is the image of

$$g(t) = \big(r(t), \theta(t), \varphi(t)\big) = \Big(R, \frac{\pi}{2}, t\Big), \qquad t \in]a, b[.$$

For technical reasons, we also consider the *trivial curve at* $x \in M$ to be a curve consisting only of the point x. Formally, this can be thought of the image of a function $\gamma: \{0\} \to M$ where $\gamma(0) = x$.

5.2.2 Submanifolds of Euclidean Space

Loosely speaking, a *submanifold* of a manifold M is a subset of M that qualifies as a manifold in its own right. For example, $\mathbb{R}^3$ is a three-dimensional manifold, and the unit sphere $S^2 \subset \mathbb{R}^3$ is a two-dimensional submanifold of $\mathbb{R}^3$. In addition, the equator $\{(x, y, z) \in S^2 : z = 0\} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\}$, which we can identify (in an obvious way) with the unit circle S^1, is a one-dimensional submanifold of S^2 (and, by transitivity, of $\mathbb{R}^3$), Thus

$$S^1 \subset S^2 \subset \mathbb{R}^3$$

where each $\subset$ denotes a submanifold inclusion. In this case, the dimension of the manifolds forms a strictly increasing sequence. However, of course, a submanifold of M^n does not *need* to be of lower dimension than M^n. For example, the interval $]-1, 1[$ is a one-dimensional submanifold of $\mathbb{R}$. More generally, the open unit ball $B^n \subset \mathbb{R}^n$ is an n-dimensional submanifold of $\mathbb{R}^n$.

A particularly important class of manifolds are those that are submanifolds of some Euclidean space $\mathbb{R}^n$. This includes the spaces $\mathbb{R}^n$ themselves, as well as the (manifold) curves and surfaces we worked with in the chapter on classical differential geometry, but it also includes many more examples, for instance, an infinite collection of 1986-dimensional submanifolds of $\mathbb{R}^{1987}$. In fact, the 'Whitney embedding theorem' (which we are not going to prove) states that every n-dimensional smooth manifold can be 'embedded' [a precise definition of an 'embedding' is not hard to give, but will not be interesting to us] in $\mathbb{R}^{2n}$. Thus, the 'particularly important class' we were talking about in fact includes the class of all smooth manifolds! Since we are essentially only interested in smooth manifolds, it follows that (essentially) all manifolds we are interested in can be thought of as submanifolds of some Euclidean space of high-enough dimension. However, it is interesting to note that it is not possible to strengthen the Whitney theorem much more. For instance, it is in general *not* possible to embed an n-dimensional manifold in $\mathbb{R}^{n+1}$.

Notice that, when we speak of some Euclidean $M = \mathbb{R}^n$ as a manifold, there is a natural choice of a coordinate system on $\mathbb{R}^n$, namely, the chart (M, ϕ) where $\phi: \mathbb{R}^n \to \mathbb{R}^n$ is the identify map on $\mathbb{R}^n$. Hence, in the case of Euclidean space (using this natural coordinate system), there is no difference between the manifold itself and its coordinate space. We implicitly used this observation in the example (↑) above.

Even though every smooth n-dimensional manifold M^n can be embedded in some Euclidean space $\mathbb{R}^m$, we still prefer to talk about such manifolds without any reference to such an $\mathbb{R}^m$. There are a number of reasons for this. First, an embedding need not be unique; thus, if you deduce a property of a manifold using some particular embedding, you don't know for sure (perhaps) whether this is an intrinsic property of the manifold, or if it only holds for this (or some)

particular embedding(s). Second, it might be difficult to find an explicit embedding of the manifold (that is, essentially, a parameterisation of the manifold considered a subset of $\mathbb{R}^m$). Third, we might not even be interested in this $\mathbb{R}^m$. For instance, if we have found out that our spacetime is some particular four-dimensional manifold M^4, then it is not obvious at all that the $\mathbb{R}^m$ with the lowest $m \geq 4$ in which M^4 can be embedded should have any physical significance. Finally, if you do not rely on the fact that every smooth manifold can be embedded in some higher-dimensional Euclidean space, then you can make your text self-contained without having to prove the Whitney theorem.

In this subsection, however, we will consider the special case in which a manifold M^n has an *explicit* embedding in some $\mathbb{R}^m$ (including the identity embedding).

5.2.2.1 Tangent Spaces and the Differential

We have already defined the tangent space of a surface $\Sigma = \mathbf{r}(D) \subset \mathbb{R}^3$ at $\mathbf{x} = \mathbf{r}(u, v)$ as the linear span of $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$, often imagined as a copy of $\mathbb{R}^2$ translated as to have its origin at $\mathbf{x} \in \mathbb{R}^3$. Although we didn't, we could also have defined the tangent space of a *curve* $\gamma = \mathbf{r}(I) \subset \mathbb{R}^n$ at $\mathbf{x} = \mathbf{r}(t)$ to be the linear span of $\mathbf{r}'(t)$, imagined as a copy of $\mathbb{R}$ with its origin at $\mathbf{x} \in \mathbb{R}^n$.

We will now generalise the tangent space to any submanifold $M^n \subseteq \mathbb{R}^m$. This is pretty easy:

Definition NN

Let $M^n \subseteq \mathbb{R}^m$ be a submanifold of Euclidean $\mathbb{R}^m$, where $n \in [1..m]$. Fix any point $x \in M$. Then the tangent space T_xM of M at x is the linear subspace of $\mathbb{R}^m$ consisting of (or, if you prefer, 'spanned by') all possible tangent vectors

$$\frac{d}{dt}\big(\mathbf{p}(t)\big)\Big|_{t=0}$$

where $\mathbf{p}\colon I \to \mathbb{R}^m$ is any differentiable f-curve *on* M [that is, $\mathbf{p}(I) \subseteq M$] passing through x at $t = 0$ [that is, $\mathbf{p}(0) = x$]. T_xM is a copy of $\mathbb{R}^n$, and we imagine that its origin lies at $x \in M^n$.

If $M^n = \mathbb{R}^n$, then the tangent space at every point $x \in M^n$ is a copy of $\mathbb{R}^n$. In particular, in $\mathbb{R}^3$, every tangent space is a copy of $\mathbb{R}^3$ with origin imagined at $\mathbf{x} \in \mathbb{R}^3$, that is, $T_{\mathbf{x}}\mathbb{R}^3$ is the set of all geometric vectors based at $\mathbf{x} \in \mathbb{R}^3$. For example, if a particle is at $\mathbf{r}(t) \in \mathbb{R}^3$ at time t, then its velocity at this time is $\dot{\mathbf{r}}(t) \in T_{\mathbf{r}(t)}\mathbb{R}^3$ which is a vector based at $\mathbf{r}(t)$.

Consider now a (differentiable) map $F\colon M^n \to N^m$ between two submanifolds of Euclidean space. A (differentiable) curve $\gamma_1\colon I \to M$ on the first manifold will induce a curve $\gamma_2 \coloneqq F \circ \gamma_1\colon I \to N$ on the second manifold, that is,

$$\gamma_2(t) = F\big(\gamma_1(t)\big), \qquad \forall t \in I.$$

Differentiation yields

$$\dot{\gamma}_2(t) = \dot{F}\big(\gamma_1(t)\big) \cdot \dot{\gamma}_1(t)$$

where $\dot{F}\big(\gamma_1(\,t)\big)$ is the Jacobian matrix of F evaluated at $\gamma_1(t)$. Notice that

$$\dot{\gamma}_1(t) \in T_{\gamma_1(t)}M, \qquad \dot{\gamma}_2(t) \in T_{\gamma_2(t)}N.$$

Thus, every (differentiable) function $F\colon M \to N$ induces a collection of linear transformations $\dot{F}_x\colon T_xM \to T_{F(x)}N$, that is, one at every point $x \in M = \text{domain}\, F$, each of which can be interpreted

as sending tangent vectors on M to the 'corresponding' (in the precise sense given above) tangent vectors on N. The matrix of such a transformation $\dot{F}_x$ is simply the Jacobian matrix of F evaluated at x. $\dot{F}_x$ is called the *differential* of F at x and is denoted F_*, where (somewhat unfortunately) the x-dependance is understood.

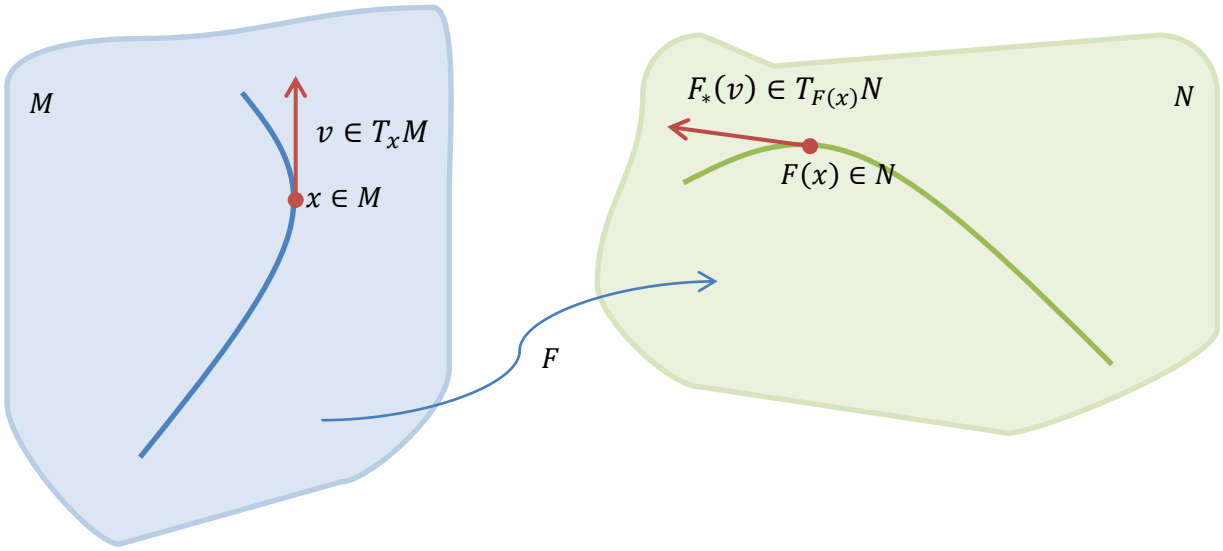


Figure 52. The differential F_* of a map $F: M \to N$ at a point $x \in M$.

We have actually encountered a number of differentials already, perhaps most notably in the context exemplified by Lemma NN. Indeed, if $\gamma = \mathbf{p}(I) = \mathbf{r}\big(\mathbf{q}(I)\big)$ is a curve on a surface $\Sigma = \mathbf{r}(D) \in \mathbb{R}^3$, then

$$\dot{\mathbf{p}}(t) = \dot{\mathbf{r}}\big(\mathbf{q}(t)\big) \cdot \dot{\mathbf{q}}(t)$$

where

$$\dot{\mathbf{q}}(t) \in T_{\mathbf{q}(t)} D$$

is a tangent vector in the tangent space $T_{\mathbf{q}(t)}D$ of the parameter plane D and

$$\dot{\mathbf{p}}(t) \in T_{\mathbf{r}(\mathbf{q}(t))} \Sigma$$

is the corresponding tangent vector in the tangent space $T_{\mathbf{r}(\mathbf{q}(t))}\Sigma$ of the surface Σ. $\mathbf{r}: D \to \Sigma$ is a differentiable map between the manifolds D and Σ, and so the tangent vectors $\dot{\mathbf{q}}(t)$ are sent to the tangent vectors $\dot{\mathbf{p}}(t)$ by the differential $\mathbf{r}_*$ with matrix $\dot{\mathbf{r}}\big(\mathbf{q}(t)\big)$ at $\mathbf{q}(t)$.

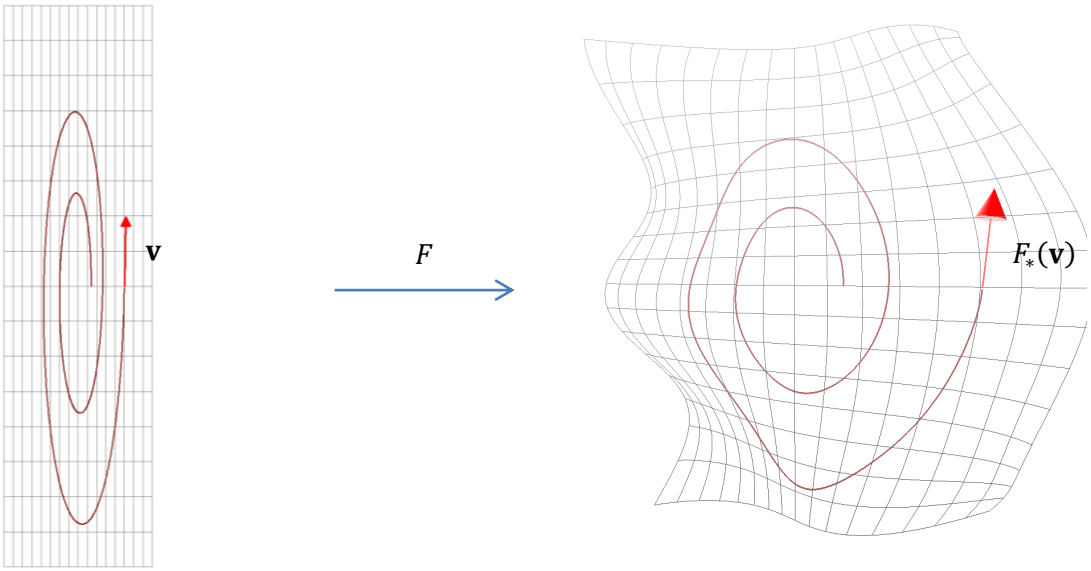


Figure 53. The differential F_* corresponding to an f-surface F.

5.2.2.2 Basis Vectors in Tangent Spaces

If we are to work efficiently with a tangent space, we need to agree on a basis in it. This is trivial in the case of $\mathbb{R}^n$. Indeed, if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are basis vectors for $\mathbb{R}^n$, then we can simply use the very same geometric vectors as the basis vectors of every tangent space $T_{\mathbf{x}}\mathbb{R}^n$ ($\mathbf{x} \in \mathbb{R}^n$). Geometrically, we simply parallel propagate the vectors from the origin of $\mathbb{R}^n$ to the origin of $T_{\mathbf{x}}\mathbb{R}^n$.

In a general manifold, this cannot be done, since there is no notion of parallel transport. In addition, although every tangent space is a copy of $\mathbb{R}^n$, where n is the dimension of the manifold, the manifold itself need not be Euclidean $\mathbb{R}^n$, and so need not to *have* a basis itself. Recall that a basis is something you can assign a vector space, and a general manifold has no vector space structure. Consider, for example, a cylinder, or a sphere. How would you define the 'sum' of two elements of such a space? Nevertheless, we will see there is a natural way of finding a basis for T_xM if $x \in U$ where (U, ϕ_U) is a coordinate chart, by using derivatives of the map ϕ_U^{-1}.

The general idea might be familiar from the language of curvilinear coordinates in ordinary calculus. For example, consider the Euclidean plane, $\mathbb{R}^2$. The usual basis vectors $\hat{\mathbf{x}} = (1, 0)$ and $\hat{\mathbf{y}} = (0, 1)$ are identified as two orthogonal, unit-length arrows starting at the origin, and these geometrical vectors also serve as the basis vectors of the tangent space $T_{\mathbf{x}}\mathbb{R}^2$ at every $\mathbf{x} \in \mathbb{R}^2$. An alternative, and more general, characterisation of these tangent-space basis vectors is obtained from the coordinate system of the plane as follows. Consider a particle at (x, y). If the particle is moved by keeping y fixed and increasing x, then it moves in the direction of $\hat{\mathbf{x}}$, and so we can *define* the tangent-space basis vector $\hat{\mathbf{x}}$ at this point as a unit vector of this direction. Similarly, $\hat{\mathbf{y}}$ may be defined as a vector of unit length pointing in the direction of motion corresponding to fixed x and increasing y. Since, at any point in the plane, a motion with increasing x and fixed y (and vice versa) occurs in the same geometrical direction, it follows that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the same geometrical vectors in every tangent space.

Now, consider instead the system of *polar* coordinates (ρ, φ), defined by the bijection

$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$

where $\rho = \sqrt{x^2 + y^2} \geq 0$ is the distance from the origin and $\varphi \in [0, 2\pi[$ is the angle that the radius vector makes with the positive x-axis. Just as every geometric point in the plane has unique coordinates (x, y), it also has unique coordinates (ρ, φ). The *coordinate curves* of any coordinate system are the curves where exactly one coordinate is increasing while the others are held fixed. Thus, the coordinate curves of the Cartesian coordinate system (x, y) are simply the images of $t \mapsto (x_0, t)$ for fixed $x_0 \in \mathbb{R}$ and $t \mapsto (t, y_0)$ for fixed $y_0 \in \mathbb{R}$ where $t \in \mathbb{R}$. As remarked above, at any point in the plane, the tangent-space basis vectors can be defined as the instantaneous unit tangent vectors of the two coordinate curves passing through the point in question. Since a general coordinate curve in polar coordinates can be parameterised

$$\mathbf{r}(\rho, \varphi) = \underline{\mathbf{e}} \begin{pmatrix} \rho \cos \varphi \\ \rho \sin \varphi \end{pmatrix},$$

keeping exactly one variable fixed, we have the tangent-space basis vectors

$$\hat{\mathbf{r}}(\rho, \varphi) \coloneqq \underline{\mathbf{e}} \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \qquad \hat{\boldsymbol{\varphi}}(\rho, \varphi) \coloneqq \underline{\mathbf{e}} \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix}$$

by differentiation and rescaling[61]. Thus, the tangent-space basis vectors, induced by the polar coordinate system, *vary* from point to point in the plane!

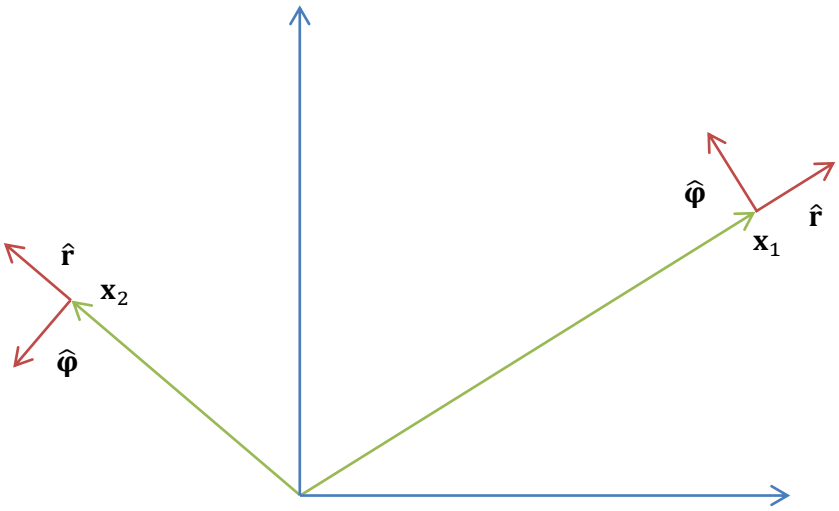


Figure 54. The tangent-space basis vectors induced by the (planar) polar coordinate system.

The idea of making use of the coordinate chart to determine tangent-space basis vectors generalises easily to general submanifolds of $\mathbb{R}^m$. Indeed, if $M^n \subseteq \mathbb{R}^m$ is a submanifold of $\mathbb{R}^m$, and $y \in M$, then there is a local parameterisation function $f: U \to M$ where $U \subseteq \mathbb{R}^n$ and $y = f(x) \in f(U)$ and the tangent vectors $\partial f / \partial x^1, \dots, \partial f / \partial x^n$ evaluated at $x = (x^1, \dots, x^n) \in U$, form a basis for the tangent space at x. In particular, if $\Sigma = \mathbf{r}(D) \subset \mathbb{R}^3$ is a surface with coordinates $(u, v) \in D$, then $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$ not only *span* the tangent space at $\mathbf{r}(u, v)$, but also serve as basis vectors in it; this is a fact we have used many times before. For the particular example of a sphere, see Figure 37 on page 199, where the basis vectors should be called $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ and stem from the usual parameterisation of a sphere.

[61] The original lengths of the derivatives are called the *scale factors* associated with the coordinates. In this case, they are 1 and ρ, respectively.

5.2.3 General Manifolds

We now turn to the case of a general manifold M^n that is *not* given as a subset of some Euclidean space. Our first goal is to define the tangent space T_xM for every $x \in M$. Perhaps the most common approach is to use the notion of a *derivation*. This approach is based on the natural identification between vectors and derivative operators known from $\mathbb{R}^n$, which we will review first.

First, let us introduce the convenient (and standard) notation

$$\partial_\alpha \coloneqq \frac{\partial}{\partial x^\alpha}$$

where $x^1, \dots, x^n$ are local coordinates on the manifold, associated with a particular coordinate patch $(U, \phi_U) \in \mathcal{A}_M$. We remark that the subscript α in ∂_α is treated like a subscript index as far as the Einstein convention is concerned.

5.2.3.1 Derivations in $\mathbb{R}^n$

Consider a scalar field $f: \mathbb{R}^n \to \mathbb{R}$ in Euclidean $\mathbb{R}^n$. If $v^a \in T_x\mathbb{R}^n$ is a *unit vector* at $x \in \mathbb{R}^n$, then the directional derivative of f at x in the direction of v^a is

$$\sum_{\alpha=1}^{n} \frac{\partial f}{\partial x^\alpha} \cdot v^\alpha = v^\alpha \partial_\alpha f,$$

evaluated at x (Cartesian coordinates assumed). In classical notation, this is the ordinary directional derivative of $f(\mathbf{x})$ at $\mathbf{x}$ in the direction of $\mathbf{v}$ (which has to be of unit length) as defined by

$$f'_{\mathbf{v}}(\mathbf{x}) = \lim_{h \to 0^+} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h} = \nabla f(\mathbf{x}) \cdot \mathbf{v}.$$

We now generalise this concept slightly by skipping the requirement that v^a is of unit length. We say that[62]

$$\mathcal{D}_v(f) \coloneqq v^\alpha \partial_\alpha f(x)$$

is the derivative of f with respect to v, an arbitrary vector, at x. This way, every vector $v \in T_x\mathbb{R}^n$ yields a unique derivative operator $\mathcal{D}_v: \mathcal{F}_x \to \mathbb{R}$, where $\mathcal{F}_x$ is the set of differentiable scalar fields defined in a neighbourhood of x. This operator satisfies two main properties.

Proposition NN

The derivative operator $\mathcal{D}_v$ is linear and Leibnitz at x, that is,

$$\mathcal{D}_v(\alpha f + \beta g) = \alpha \mathcal{D}_v(f) + \beta \mathcal{D}_v(g)$$

and

$$\mathcal{D}_v(f \cdot g) = f(x) \cdot \mathcal{D}_v(g) + \mathcal{D}_v(f) \cdot g(x)$$

for all $f, g \in \mathcal{F}_x$ and constants $\alpha, \beta \in \mathbb{R}$.

Proof

$$\mathcal{D}_v(\alpha f + \beta g) = v^\gamma \partial_\gamma(\alpha f + \beta g) = \alpha v^\gamma \partial_\gamma(f) + \beta v^\gamma \partial_\gamma(g) = \alpha \mathcal{D}_v(f) + \beta \mathcal{D}_v(g)$$

[62] In the symbol $\mathcal{D}_v(f)$ the dependence on the point $x \in M$ is implied because the vector $v \in T_xM$.

and

$$\mathcal{D}_v(f\cdot g) = v^\alpha\partial_\alpha(f\cdot g) = v^\alpha\big(f(x)\cdot\partial_\alpha g + \partial_\alpha f\cdot g(x)\big) = f(x)\cdot v^\alpha\partial_\alpha(g) + g(x)\cdot v^\alpha\partial_\alpha(f) = \\ = f(x)\cdot\mathcal{D}_v(g) + \mathcal{D}_v(f)\cdot g(x).$$

■

Definition NN

An operator $\mathcal{D}\colon\mathcal{F}_x\to\mathbb{R}$ that is linear and Leibniz at x is called a *derivation* at $x\in\mathbb{R}^n$.

Conversely, one can show that every derivation at $x\in\mathbb{R}^n$ is of the form (↑) for some unique vector $v^a\in T_x\mathbb{R}^n$. Hence, (↑) displays a bijection from the tangent space $T_x\mathbb{R}^n$ to the set of derivations at x.

5.2.3.2 Derivations in General Manifolds

Although the concept of a tangent space, as defined using an ambient Euclidean space (as in Definition NN), cannot be translated immediately to general manifolds (the best way to see that is to attempt and fail), it turns out that the concept of derivations can. In comparison, this is rather straightforward:

Definition NN

Let M^n be a manifold, and let $x\in M$. Then $\mathcal{F}_x$ is the set of real-valued differentiable functions defined in some neighbourhood of x, and an operator $\mathcal{D}\colon\mathcal{F}_x\to\mathbb{R}$ that is linear and Leibniz at x is called a *derivation* at x. Finally, $\mathcal{X}_xM$ is the set of all derivations at x.

(Definition and) Proposition NN

Let $\mathcal{D},\mathcal{D}_1,\mathcal{D}_2\in\mathcal{X}_xM$ be any three derivations, and let $\alpha\in\mathbb{R}$ be a constant. Let $(\mathcal{D}_1+\mathcal{D}_2)\colon\mathcal{F}_x\to\mathbb{R}$ and $(\alpha\mathcal{D})\colon\mathcal{F}_x\to\mathbb{R}$ be defined by $(\mathcal{D}_1+\mathcal{D}_2)(f)\coloneqq\mathcal{D}_1(f)+\mathcal{D}_2(f)$ and $(\alpha\cdot\mathcal{D})(f)\coloneqq\alpha\mathcal{D}(f)$, respectively, for all $f\in\mathcal{F}_x$. Then $(\mathcal{X}_xM,+,\cdot)$ is a vector space.

Proof

By definition, for any pair of functions $f,g\in\mathcal{F}_x$ and constants $\alpha,\beta\in\mathbb{R}$ we have

$$(\mathcal{D}_1+\mathcal{D}_2)(\alpha f+\beta g) = \mathcal{D}_1(\alpha f+\beta g)+\mathcal{D}_2(\alpha f+\beta g) = \\ = \alpha\mathcal{D}_1(f)+\beta\mathcal{D}_1(g)+\alpha\mathcal{D}_2(f)+\beta\mathcal{D}_2(g) = \\ = \alpha\mathcal{D}_1(f)+\alpha\mathcal{D}_2(f)+\beta\mathcal{D}_1(g)+\beta\mathcal{D}_2(g) = \alpha(\mathcal{D}_1+\mathcal{D}_2)(f)+\beta(\mathcal{D}_1+\mathcal{D}_2)(g);$$

thus, $(\mathcal{D}_1+\mathcal{D}_2)$ is linear. Furthermore,

$$(\mathcal{D}_1+\mathcal{D}_2)(f\cdot g) = \mathcal{D}_1(f\cdot g)+\mathcal{D}_2(f\cdot g) = \\ = f(x)\cdot\mathcal{D}_1(g)+\mathcal{D}_1(f)\cdot g(x)+f(x)\cdot\mathcal{D}_2(g)+\mathcal{D}_2(f)\cdot g(x) = \\ = f(x)\cdot(\mathcal{D}_1+\mathcal{D}_2)(g)+(\mathcal{D}_1+\mathcal{D}_2)(f)\cdot g(x),$$

and so $(\mathcal{D}_1+\mathcal{D}_2)$ is Leibniz at x, too. Therefore, $(\mathcal{D}_1+\mathcal{D}_2)\in\mathcal{X}_xM$. The proof that $(\alpha\mathcal{D})\in\mathcal{X}_xM$ is analogous (but easier). The rest of the vector space axioms are trivially satisfied. ■

Proposition NN

If $M^n\subseteq\mathbb{R}^m$ is a submanifold of Euclidean space, then $\mathcal{X}_xM$ is isomorphic to T_xM for every $x\in M$, where the isomorphism is

$$T_xM \ni v \leftrightarrow \mathcal{D}_v \in \mathcal{X}_xM$$

where

$$\mathcal{D}_v(f) \stackrel{\text{def}}{=} v^\alpha \partial_\alpha f(x).$$

Proof

For every $f \in \mathcal{F}_x$,

$$\mathcal{D}_{u+v}(f) \stackrel{\text{def}}{=} (u^\alpha + v^\alpha)\partial_\alpha f(x) = u^\alpha \partial_\alpha f(x) + v^\alpha \partial_\alpha f(x) \stackrel{\text{def}}{=} \mathcal{D}_u(f) + \mathcal{D}_v(f) \stackrel{\text{def}}{=} (\mathcal{D}_u + \mathcal{D}_v)(f)$$

and so the mapping from vectors to differential operators is linear, that is, a vector space homomorphism. The rest of the proof is left as an exercise. ■

In the case where M is a submanifold of Euclidean space, $\mathcal{X}_xM$ is isomorphic to the tangent space T_xM, where the isomorphism is given by (↑). It is therefore natural to make

Definition NN

Let M^n be a manifold, and let $x \in M$. The *tangent space* T_xM of M at x is the space $\mathcal{X}_xM$ of derivations at x. Each element of T_xM is called a *tangent vector* at x.

A *vector field* is an assignment of a tangent vector to each tangent space of a manifold. In order to specify tangent vectors, and thus vector fields, we need to decide on a basis in each tangent space.

Proposition NN

Consider a tangent space T_yM and a coordinate chart $(U, \phi_U) \in \mathcal{A}_{M,y}$ where y has coordinates $x^\alpha = \phi_U(y^\alpha)$. Then the tangent vectors

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^1}}, \dots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^n}}$$

defined by

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^\alpha}} \coloneqq \left(f \mapsto \left[\partial_\alpha \left(f(\phi_U^{-1}(x'))\right]\right]_{x'=x}\right), \qquad \forall f \in \mathcal{F}_y$$

form a basis for T_yM.

Proof

TBW.

Notice that, according to the 'coordinate patch observation', the expression $\frac{\partial}{\partial x^\alpha}(f \circ \phi_U^{-1})$ is simply the derivative of f in the direction of x^α. In the case of a manifold embedded in $\mathbb{R}^m$, the basis $\boldsymbol{\partial_1}, \dots, \boldsymbol{\partial_n}$ [as we will often abbreviate $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^1}}, \dots, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^n}}$] reduces to the usual tangent-space basis. Indeed, if $\Sigma = \mathbf{r}(D) \subset \mathbb{R}^3$ is a manifold surface covered by a single coordinate patch (Σ, ϕ), then $\phi^{-1} = \mathbf{r}$, as usual. Thus, with coordinates $x^1 = u$ and $x^2 = v$,

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^\alpha}}(f) = \frac{\partial}{\partial x^\alpha}\left(f(\phi^{-1}(x^1, x^2)\right) = \frac{\partial}{\partial x^\alpha}\Big(f\big(\mathbf{r}(x^1, x^2)\big)\Big) = \nabla f \cdot \frac{\partial \mathbf{r}}{\partial x^\alpha}$$

for any $f \in \mathcal{F}_{\mathbf{r}(x^1,x^2)}$. Using the isomorphism between derivations and vectors in Euclidean space, we see that the two basis vectors are

$$\frac{\partial \mathbf{r}}{\partial x^1} = \frac{\partial \mathbf{r}}{\partial u}$$

and

$$\frac{\partial \mathbf{r}}{\partial x^2} = \frac{\partial \mathbf{r}}{\partial v}$$

as expected. The basis $\boldsymbol{\partial_1}, \dots, \boldsymbol{\partial_n}$ is called the *coordinate* basis, and clearly depends upon the coordinate patch used.

5.2.3.3 The Cotangent Space

Definition NN

Let M be a manifold. The *cotangent space* $T_x^*M \coloneqq (T_xM)^*$ at $x \in M$ is the dual space of the tangent space T_xM at x. Elements of $(T_xM)^*$ are called *dual vectors*, *covariant vectors*, or simply *covectors*.

We have already seen that there is a natural choice of basis in a dual space given a basis in the original space. Indeed, the ith dual-space basis vector is the linear functional that reads off the ith coordinate of a vector. In this subsection, we will investigate the relation between this basis and the coordinate chart of the manifold.

In Section 5.2.2.1 we introduced the differential of a map $F\colon \mathbb{R}^n \to \mathbb{R}^m$. We now make

Definition NN

Let $f\colon M \to \mathbb{R}$. The *differential* of f at $y \in M$ is the linear functional

$$df(v) \coloneqq \mathcal{D}_v(f) \coloneqq v^\alpha(\partial_\alpha f)(y), \qquad \forall v \in T_yM$$

where the partial derivatives are with respect to a coordinate chart $(U, \phi_U) \in \mathcal{A}_{M,y}$.

Strictly speaking, $df(v) = v^\alpha\left(\partial_\alpha(f \circ \phi_U^{-1})\right)(x)$. It is easily verified that the expression $v^\alpha\partial_\alpha f$ has the same value in any coordinate system, despite the fact that the differential operators ∂_α themselves are (obviously) coordinate system dependent.

If $M = M^n$ is n-dimensional, then we may consider the differentials of the n coordinate functions $x^1, \dots, x^n$. By definition, these are

$$dx^\alpha(v) = \mathcal{D}_v(x^\alpha) = v^\beta\partial_\beta x^\alpha = v^\beta\delta_\beta^\alpha = v^\alpha;$$

thus, dx^α *reads off the αth component of its argument!* That is,

$$dx^\alpha = \boldsymbol{\sigma}^\alpha$$

where $\boldsymbol{\sigma}^\alpha$ are the dual basis vectors. In other words, the covectors

$$dx^1, \dots, dx^n$$

form a basis for the cotangent space $(T_xM)^*$ of M at $x \in M$ that is adapted to the local coordinate chart.

Observation NN

A local coordinate chart $(U, \phi_U) \in \mathcal{A}_M$ yields a basis $\boldsymbol{\partial_1}, \ldots, \boldsymbol{\partial_n}$ in every tangent space T_xM and a basis $dx^1, \ldots, dx^n$ in every cotangent space T_x^*M ($x \in U \subseteq M^n$).

5.3 Tensor Fields on Manifolds

In this section, we will introduce the concept of *tensor fields* on manifolds.

5.3.1 Tensor Fields

5.3.1.1 Vector Fields

A *vector field* on a (subset of a) manifold is a rule that singles out a tangent vector in each tangent space of the (subset of the) manifold; as is common practice, we also refer to the image of this rule as the 'vector field'. A vector field on a general manifold M is, naturally, specified using a local coordinate system $(U, \phi_U) \in \mathcal{A}_M$. Thus, it is of the form

$$F(y) = F^\alpha(y)\boldsymbol{\partial_\alpha}, \qquad y \in M$$

where each $F^\alpha: M \to \mathbb{R}$. In practice, we specify also the points in M using the coordinate chart, and write

$$G(x) = G^\alpha(x)\boldsymbol{\partial_\alpha}, \qquad x \in \phi_U(U) \subseteq \mathbb{R}^n$$

where each $G^\alpha: \phi_U(U) \to \mathbb{R}$ and $y = \phi_U^{-1}(x)$. Clearly, the coordinate basis vectors $\boldsymbol{\partial_\alpha}$ are themselves examples of vector fields. A vector field F is said to be *smooth* (continuous, differentiable, ...) iff the functions $G^\alpha: \phi_U(U) \to \mathbb{R}$ are smooth (continuous, differentiable, ...), as is obvious from the 'coordinate patch observation'.

For example, a vector field on the usual two-sphere S^2 is of the form

$$G(\theta, \varphi) = f(\theta, \varphi)\boldsymbol{\partial_\theta} + g(\theta, \varphi)\boldsymbol{\partial_\varphi},$$

or, in classical notation,

$$\mathbf{G}(\theta, \varphi) = f(\theta, \varphi)\hat{\boldsymbol{\theta}} + g(\theta, \varphi)\hat{\boldsymbol{\varphi}}.$$

5.3.1.2 Transformation Properties of Tangent Vectors

Since a tangent vector *is* a vector in a tangent vector space, it is a contravariant tensor of rank 1 over this space, and transforms as such under a change of basis *in this space*. The question is, "how does the tangent-space basis vectors actually change if you change coordinates in the manifold?" [Recall Observation NN, which states that the coordinate basis of each tangent space is induced by the manifold's coordinate chart.]

To investigate this, let $(U, \phi_U) \in \mathcal{A}_{M,y}$ and $(V, \phi_V) \in \mathcal{A}_{M,y}$ be two charts containing the point $y \in M$. Let $f_{VU} \coloneqq \phi_V \circ \phi_U^{-1}: \phi_U(U \cap V) \to \phi_V(U \cap V)$ be the transition function. If x^α are coordinates in U and x'^α are coordinates in V, then the Jacobian of the transition function is

$$\dot{f}_{UV} = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \cdots & \frac{\partial x'^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x'^n}{\partial x^1} & \cdots & \frac{\partial x'^n}{\partial x^n} \end{pmatrix}.$$

If the T_yM coordinate basis vectors are denoted $\boldsymbol{\partial_\alpha}$ and $\boldsymbol{\partial'_\alpha}$ when induced by each coordinate patch, then, by the chain rule,

$$\boldsymbol{\partial_\beta} = \frac{\partial x'^\alpha}{\partial x^\beta} \cdot \boldsymbol{\partial'_\alpha}$$

which completely describes the vector-space change of basis. Comparing with Equation NN on page NN, we have

$$C^{-1} = \dot{f}_{UV}.$$

Thus, the vector transformation rule,

$$v'^{\alpha} = (C^{-1})^{\alpha}_{\beta} v^{\beta},$$

can be written

$$\boxed{v'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} v^{\beta}.}$$

5.3.1.3 Covector Fields

Similarly, a *covector field* on a manifold is a rule that selects a single covector from each covector space in the manifold. A covector field on a general manifold M is, naturally, specified using a local coordinate system $(U, \phi_U) \in \mathcal{A}_M$. Thus, it is of the form

$$F(y) = F_{\alpha}(y) dx^{\alpha}, \qquad y \in M$$

where each $F^{\alpha}: M \to \mathbb{R}$. In practice, we specify also the points in M using the coordinate chart, and write

$$G(x) = G_{\alpha}(y) dx^{\alpha}, \qquad x \in \phi_U(U) \subseteq \mathbb{R}^n$$

where each $G^{\alpha}: \phi_U(U) \to \mathbb{R}$ and $y = \phi_U^{-1}(x)$. The basis vectors dx^{α} are themselves examples of covector fields. A covector field F is said to be *smooth* iff the functions $G_{\alpha}: \phi_U(U) \to \mathbb{R}$ are smooth.

5.3.1.4 Transformation Properties of Covectors

Using the results from Section 5.3.1.2, we have that a covector transforms like

$$\boxed{v'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} v_{\beta}}$$

under a change from unprimed to primed coordinates in the manifold.

5.3.1.5 Tensor Fields

More generally, a tensor field on a manifold M is a rule that, at every point $x \in M$ picks out a tensor over the tangent space $T_x M$ at x. As usual, we specify a tensor field using a local coordinate system on M. For example, a covariant tensor field of rank 2 will be written

$$T(y) = T_{\alpha\beta}(y) dx^{\alpha} \otimes dx^{\beta}, \qquad y \in M$$

where each $T_{\alpha\beta}: M \to \mathbb{R}$, or, in practice,

$$S(x) = S_{\alpha\beta}(x) dx^{\alpha} \otimes dx^{\beta}, \qquad x \in \phi_U(U) \subset \mathbb{R}^n$$

where each $S_{\alpha\beta}: \phi_U(U) \to \mathbb{R}$ and $y = \phi_U^{-1}(x)$.

5.3.1.6 The Classical Definition of a Tensor

We will now (finally!) be able to explain the term 'almost' that we (mis-) used in Section 5.1.7.6. The thing is, that in classical tensor analysis and physics text, one usually means 'tensor *field*' when one speaks simply of a 'tensor'. Thus, such a text would probably do

Classical Definition NN

A *contravariant tensor of rank* k is a set of n^k scalar fields $a^{\alpha_1,\ldots,\alpha_k}$ $(\alpha_i \in [1..n])$ such that, under a change of basis described by the matrix C, they transform as

$$(a')^{\alpha_1,\ldots,\alpha_k} = \frac{\partial x'^{\alpha_1}}{\partial x^{\beta_1}} \cdots \frac{\partial x'^{\alpha_k}}{\partial x^{\beta_k}} a^{\beta_1,\ldots,\beta_n}.$$

A *covariant tensor of rank* k is a set of n^k scalar fields $a_{\alpha_1,\ldots,\alpha_k}$ $(\alpha_i \in [1..n])$ such that, under a change of basis, they transform as

$$a'_{\alpha_1,\ldots,\alpha_k} = \frac{\partial x^{\beta_1}}{\partial x'^{\alpha_1}} \cdots \frac{\partial x^{\beta_k}}{\partial x'^{\alpha_k}} a_{\beta_1,\ldots,\beta_2}.$$

A tensor of type (k, l) and rank $k + l$ is a set of n^{k+l} scalar fields $a^{\alpha_1,\ldots,\alpha_k}_{\beta_1,\ldots,\beta_k}$ $(\alpha_i, \beta_i \in [1..n])$ such that, under a change of basis, they transform as

$$(a')^{\alpha_1,\ldots,\alpha_k}_{\beta_1,\ldots,\beta_k} = \frac{\partial x'^{\alpha_1}}{\partial x^{\gamma_1}} \cdots \frac{\partial x'^{\alpha_k}}{\partial x^{\gamma_k}} \frac{\partial x^{\delta_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{\delta_k}}{\partial x'^{\beta_k}} a^{\gamma_1,\ldots,\gamma_k}_{\delta_1,\ldots,\delta_k}$$

where $k, l \in \mathbb{N}$ and the tensor is said to be *mixed* if $k, l > 0$.

5.3.2 Riemannian Manifolds

We have seen that, in any vector space, an inner product can be described by a covariant tensor, called the metric tensor. We will now define a smooth tensor field, which assigns such a tensor to each tangent space in a manifold. We first make

Definition NN

Consider tensors over a vector space X. A tensor T_{ab} is *positive definite* iff $T_{ab}(\mathbf{u}, \mathbf{u}) \geq 0$ and $T_{ab}(\mathbf{u}, \mathbf{u}) = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$, for all $\mathbf{u} \in X$. A tensor T_{ab} is non-degenerate iff $T_{ab}(\mathbf{u}, \mathbf{v}) = 0, \forall \mathbf{u} \in X \Rightarrow \mathbf{v} = \mathbf{0}$.

Corollary NN

A positive definite tensor is non-degenerate.

Proof

Let T_{ab} be positive definite, and assume that $T_{ab}(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{u} \in X$. Then, in particular, $T_{ab}(\mathbf{v}, \mathbf{v}) = 0$ and so $\mathbf{v} = \mathbf{0}$ by positive-definiteness. ■

Naturally, a positive-definite (non-degenerate) tensor *field* is a tensor field such that at every point $x \in M$, the tensor over $T_x M$ is positive definite (non-degenerate). We now define a metric tensor field, and we also give a special name to manifolds equipped with such a field.

Definition NN

A *Riemannian manifold* is a smooth manifold with a prescribed smooth, symmetric, and positive-definite tensor field g_{ab}. A *pseudo-Riemannian manifold* is a smooth manifold with a prescribed

smooth, symmetric, and non-degenerate tensor field g_{ab}. In each case, the tensor field g_{ab} is called the *metric [tensor [field]]* of the manifold.

Notice that a Riemannian manifold is a manifold in which every tangent space has a metric tensor, that is, essentially, an inner product. A pseudo-Riemannian manifold is a slightly more general object, in which every tangent space has a bilinear function that satisfies some of the axioms for an inner product (such as symmetry or 'commutativity'), but not the axiom of positive-definiteness. This means that a tangent vector in a pseudo-Riemannian manifold may have negative norm-squared, where the norm-square $\|\cdot\|^2$, as usual, is induced by the inner product $\langle\cdot,\cdot\rangle$ by

$$\|v\|^2 \coloneqq \langle v, v\rangle \coloneqq g_{ab}v^a v^b, \qquad \forall v \in T_x M.$$

A two-dimensional (manifold) surface $\Sigma \subset \mathbb{R}^3$ is a Riemannian manifold with the first fundamental form as its metric tensor. More precisely, the metric tensor $g_{\mathbf{x}}: T_{\mathbf{x}}\Sigma \times T_{\mathbf{x}}\Sigma \to \mathbb{R}$ of Σ at $\mathbf{x} \in \Sigma$ is defined by

$$g_{\mathbf{x}}(\mathbf{u}, \mathbf{v}) = \mathbf{v}^T \mathcal{F} \mathbf{u}, \qquad \forall \mathbf{u}, \mathbf{v} \in T_{\mathbf{x}}\Sigma$$

where $\mathcal{F}$ is the first fundamental form of Σ. Being a quadratic form, $g_{\mathbf{x}}$ is symmetric, and, in addition, it is positive-definite since $\mathcal{F}$ is; hence, $g_{\mathbf{x}}$ is indeed a metric tensor. Usually, we will write it simply as g or g_{ab}, where it is understood that it is a tensor *field*.

Using a local coordinate chart $(U, \phi) \in \mathcal{A}_M$, (the image of) a metric on a (pseudo-) Riemannian manifold is written

$$g_{ab}(\mathbf{u}, \mathbf{v}) = g_{\alpha\beta}(y) u^\alpha v^\beta, \qquad \forall \mathbf{u}, \mathbf{v} \in T_y M$$

where $y \in M$ or, in practice,

$$g_{ab}(\mathbf{u}, \mathbf{v}) = g_{\alpha\beta}(x) u^\alpha v^\beta, \qquad \forall \mathbf{u}, \mathbf{v} \in T_y M$$

where $x = \phi(y)$. Thus, the tensor *itself* may be written

$$g_{ab} = g_{\alpha\beta} dx^\alpha \otimes dx^\beta,$$

where $dx^1, \dots, dx^n$ is the dual basis field of the chart. Usually we omit the tensor multiplication sign and denote the tensor by ds^2. Then we end up with

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta.$$

In addition, we write

$$(dx^\alpha)^2 \coloneqq dx^\alpha dx^\alpha$$

and if our local coordinates are $x^1 = x$, $x^2 = y$, and $x^3 = z$, say, then we may even use the dangerous notation

$$dx^2 \coloneqq (dx)^2 \overset{\text{def}}{=} (dx^1)^2 \overset{\text{def}}{=} dx^1 dx^1 = dx dx$$

and similarly for dy^2 and dz^2. The reason for these conventions should be apparent from the following example.

Example NN

In classical physics, space is a three-dimensional Riemannian manifold, namely, $M = \mathbb{R}^3$ equipped with the (constant) metric

$$ds^2 = dx^2 + dy^2 + dz^2$$

where x, y, and z are Cartesian coordinates in M. Now, let $v^a \in T_pM$ at some $p \in M$. Then the norm-squared of v^a is

$$\begin{aligned} ds^2(v^a, v^b) = g_{ab}v^a v^b = (dx^2 + dy^2 + dz^2)(v^a, v^b) = \\ = (dx \otimes dx)(v^a, v^b) + (dy \otimes dy)(v^a, v^b) + (dz \otimes dz)(v^a, v^b) = \\ = dx(v^a) \cdot dx(v^b) + dy(v^a) \cdot dy(v^b) + dz(v^a) \cdot dz(v^b) = \\ = v^1 \cdot v^1 + v^2 \cdot v^2 + v^3 \cdot v^3 = (v^1)^2 + (v^2)^2 + (v^3)^2. \end{aligned}$$

Thus, the expression (↑) for a metric simply says that the norm-square of a vector is the sum of the norm-squares of the components of the vector! In classical tensor analysis, one would say, slightly less rigorously, that ds is the length of an 'infinitesimally small displacement' corresponding to (equally infinitesimal) changes dx, dy, and dz in the coordinates x, y, and z.

Notice that the components

$$g_{\alpha\beta} \equiv g_{ab}(\mathbf{e}_\alpha)^a(\mathbf{e}_\beta)^b = \delta^\alpha_\beta$$

so we may write

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5.3.2.1 Curves

Consider a smooth curve $\gamma: I \to M$ on a manifold M^n. To each point $t \in I$, we wish to assign a tangent vector $\dot\gamma \in T_{\gamma(t)}M$ to the curve. This is rather straightforward, even in the most general case.

Definition NN

Let $\gamma: I \to M$ be a smooth curve on a smooth manifold M^n. Then the *tangent vector* to the curve at $t_0 \in I$ is the tangent vector $T: \mathcal{F}_{\gamma(t_0)} \to \mathbb{R} \in T_{\gamma(t_0)}M$ defined by

$$T(f) \coloneqq \frac{d}{dt}\big(f(\gamma(t))\big|_{t=t_0}, \qquad \forall f \in \mathcal{F}_{\gamma(t_0)}$$

where the derivative is taken in the ordinary sense, for $f \circ \gamma: \mathbb{R} \to \mathbb{R}$.

Now, pick a coordinate chart (U, ϕ) for the manifold. Then

$$T(f) = \frac{d}{dt}\big(f(\gamma(t))\big|_{t=t_0} = \frac{d}{dt}\Big(f\big(\phi^{-1}\big(\phi(\gamma(t))\big)\big)\Big|_{t=t_0} = \frac{d}{dt}\big(g(\psi(t))\big)\Big|_{t=t_0}$$

where $g \coloneqq f \circ \phi^{-1}: \mathbb{R}^n \to \mathbb{R}$ is the coordinate expression for f, and $\psi \coloneqq \phi \circ \gamma: I \to \mathbb{R}^n$ is the coordinate expression for the curve (thus, a curve in coordinate space). But then

$$T(f) = \nabla g \cdot \dot\psi = \dot\psi^\alpha \partial_\alpha g = \dot\psi^\alpha \partial_\alpha f\big(\phi^{-1}(t)\big) = \dot\psi^\alpha \boldsymbol{\partial_\alpha}(f)$$

using the definition of the coordinate basis. Thus, the components of the curve's tangent vector in the coordinate basis are $\dot{\psi}^\alpha$, that is, the coordinates of the usual coordinate-space curve $\phi \circ \gamma \colon I \to \mathbb{R}^n$. It also follows from this that the vectors $\boldsymbol{\partial_a}$ are the tangent vectors to the corresponding coordinate curves.

We can now define the *length* of an arbitrary curve on a Riemannian manifold:

Definition NN

Let $\gamma \colon I \to M$ be a smooth curve on a smooth Riemannian manifold M^n. Then the *length* of γ is

$$L_\gamma \coloneqq \int_\gamma ds \coloneqq \int_a^b \sqrt{ds^2(\dot{\gamma},\dot{\gamma})}dt = \int_a^b \sqrt{\left(ds^2\left(\gamma(t)\right)\right)\left(\dot{\gamma}(t),\dot{\gamma}(t)\right)}\,dt = \int_a^b \sqrt{g_{cd}\dot{\gamma}^c\dot{\gamma}^d}dt$$

where $a = \inf I$ and $b = \sup I$.

If $M = M^2 \subset \mathbb{R}^3$ is a manifold surface with its first fundamental form as its metric tensor field, then the length of any curve on M, as given by the previous definition (which holds for *any* Riemannian manifold) coincides with the length as computed using classical theory (cf. Section 4.3.4.1). But be sure to notice that our new machinery is *far* more general!

Example NN

Let $M = \mathbb{R}^2$ be the Euclidean plane with Cartesian coordinates and (hence) metric

$$ds^2 = dx^2 + dy^2$$

and consider the unit circle

$$\gamma(t) = (\cos t\,, \sin t), \qquad t \in \,]0, 2\pi[.$$

The metric has components

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the coordinates of the curve's tangent vector are

$$\dot{\gamma}^\alpha = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

at time t. Thus, the length of the curve is

$$L_\gamma \overset{\text{def}}{=} \int_0^{2\pi} \sqrt{g_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta}\,dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\,dt = 2\pi.$$

Example NN

Again, let $M = \mathbb{R}^2$ but instead of Cartesian coordinates (x, y), let us employ polar coordinates (r, φ), as defined by $x = r\cos\varphi\,, y = r\sin\varphi$. Formally, let $(\mathbb{R}^2, \phi)$ be the Cartesian coordinate

patch, where $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ is the identity function, and let $(\mathbb{R}^2, \psi)$ be the polar coordinate patch. The metric tensor has components

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the Cartesian system, and so, in the polar system, the components of the metric tensor are

$$g'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta} g_{\gamma\delta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

which is written

$$ds^2 = dr^2 + r^2 d\varphi^2$$

in classical notation. Now, the same curve as in the last example (i.e., the unit circle) is

$$\gamma(t) = (1, t), \qquad t \in]0, 2\pi[$$

and therefore the components of the curve's tangent vector are

$$\dot{\gamma}'^\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, the length of the curve is

$$L_\gamma \stackrel{\text{def}}{=} \int_0^{2\pi} \sqrt{g'_{\alpha\beta}\dot{\gamma}'^\alpha \dot{\gamma}'^\beta}\, dt = \int_0^{2\pi} \sqrt{r^2}\, dt = \int_0^{2\pi} 1 dt = 2\pi.$$

Example NN

Let M^2 be a Riemannian manifold with coordinates (u, v) in $\mathbb{R}^2$, and metric

$$ds^2 = du^2 + e^u dv^2.$$

Consider the curve $\gamma(t) = (t, t)$ with $t \in]-5, 5[$. We want to find the length of the curve. But we have already computed this curve length, in Example NN, where we formulated the problem in terms of the first fundamental form of a surface in 3-space!

Example NN

Let M^3 be a Riemannian manifold with coordinates (x, y, z) in $\mathbb{R}^3$, and metric

$$ds^2 = e^{2x}\, dx^2 + (1 + y^2)\, dy^2 + dz^2.$$

Consider the curve $\gamma(t) = (t, e^t, t)$, $t \in]-1, 1[$. The length of this curve is

$$L_\gamma \stackrel{\text{def}}{=} \int_{-1}^{1} \sqrt{e^{4t} + 2e^{2t} + 1}\, dt = \int_{-1}^{1} (e^{2t} + 1) dt = 2 + \sinh 2 \approx 5.6.$$

A Riemannian manifold has a positive definite metric tensor that we use, for example, to find the length of curves. This, in turn, allows us to define a metric (in the algebra sense of the word) on the manifold.

Definition NN

Let M be a Riemannian manifold with metric tensor ds^2. Define

$$d(x,y) \coloneqq \inf_{\gamma\in\Gamma} L_\gamma = \inf_{\gamma\in\Gamma} \int_\gamma \sqrt{ds^2}$$

where Γ is the set of all piecewise smooth curves starting at $x \in M$ and ending at $y \in M$.

Proposition NN

The function $d\colon M \times M \to \mathbb{R}$ is an extended metric on M.

Proof

For all $x, y, z \in M$, we have

1. $d(x,y) \in [0,\infty]$

   This is trivial, because the integrand is non-negative.

2. $d(x,y) = 0 \Leftrightarrow x = y$

   We regard the trivial curve at $x \in M$ to be 'piecewise smooth'. ⇒) If $x = y$, then the trivial curve at x is a p.w. smooth curve from x to y, and the length of this curve is $\int_0^0 ds = 0$. Thus $d(x,y) = 0$. ⇐) TBW.

3. $d(x,y) = d(y,x)$

   The set Γ of curves from x to y is identical to the set of curves from y to x except for the orientation of the curves.[63] This affects the tangent vectors to the curve at each point, which are replaced by their vector-space inverses. Since the metric tensor is a *quadratic* form in these tangent vectors, this does not affect the image of the metric tensor.[64]

4. $d(x,z) \leq d(x,y) + d(y,z)$

   Let $\gamma_{xy}\colon [0,1] \to M$ be a p.w. smooth curve from x to y, and let $\gamma_{yz}\colon [0,1] \to M$ be such a curve from y to z. Then $\gamma\colon [0,1] \to M$ defined by

$$\gamma(t) = \begin{cases} \gamma_{xy}(2t), & t \leq \frac{1}{2} \\ \gamma_{yz}\left(2\left(t-\frac{1}{2}\right)\right), & t > \frac{1}{2} \end{cases}$$

[63] Indeed, if $\gamma\colon [0,1] \to M$ is a curve from x to y, then $\gamma'\colon [0,1] \to M$ defined by $\gamma'(t) = \gamma(1-t)$ is a curve from y to x with the same image.

[64] Formally, $g_{ab}(-u,-u) = -g_{ab}(u,-u) = -\big(-g_{ab}(u,u)\big) = g_{ab}(u,u)$ by linearity (specifically, homogeneity) in each slot.

is a curve from x to z with length $d(x,y)+d(y,z)$. If γ_{xy} and γ_{yz} are piecewise smooth, so is γ. Thus, the infimum of the lengths as taken for *all* piecewise smooth curves between x and z must be equal to this value or be smaller. ■

5.4 Differentiation of Tensor Fields

5.4.1 Push-Forwards and Pull-Backs

We are already familiar with the concept of the differential F_* of a map $F: \mathbb{R}^n \to \mathbb{R}^m$ as a linear map sending tangent vectors at $T_x\mathbb{R}^n$ to the 'corresponding' tangent vectors at $T_{F(x)}\mathbb{R}^m$. We will now generalise this to arbitrary manifolds. The underlying idea is that F very naturally 'pulls backs' functions $f: N \to \mathbb{R}$ to functions $f \circ F: M \to \mathbb{R}$.

Definition NN

Let $F: M^n \to N^m$ be a differentiable map between manifolds M^n and N^m. Then the *differential*, or *push-forward*, F_* of F at $x \in M$ is the map $F_*: T_xM \to T_{F(x)}N$ defined by

$$(F_*v)(f) = v(f \circ F), \qquad \forall v \in T_xM$$

for all $f \in \mathcal{F}_{F(x)}$.

This reduces to the familiar Euclidean concept of the differential. Indeed, if $F: \mathbb{R}^n \to \mathbb{R}^m$ and $v \in T_x\mathbb{R}^n$, and if we let $w \coloneqq F_*v$ then

$$w^\alpha \partial_\alpha f = (F_*v)(f) \overset{\text{def}}{=} v(f \circ F) = v^\alpha \partial_\alpha (f \circ F) = v^\alpha \big(\partial_\beta f \cdot \partial_\alpha F^\beta\big) = \partial_\alpha F^\beta \cdot v^\alpha \partial_\beta f.$$

But $\partial_\alpha F^\beta$ are the components of the Jacobian matrix $\dot{F}$. Thus, in classical matrix notation,

$$\mathbf{w} \cdot \nabla f = \dot{F}\mathbf{v} \cdot \nabla f.$$

Since this should hold for *any* function f, we end up with

$$\mathbf{w} = \dot{F}\mathbf{v}.$$

Hence, given a map $F: M^n \to N^m$ we obtain a map, $F_*: T_xM \to T_{F(x)}N$ sending vectors 'forwards'. But we also obtain, in a natural manner, a map $F^*: T^*_{F(x)}N \to T^*_xM$ sending covectors 'backwards':

Definition NN

Let $F: M^n \to N^m$ be a differentiable map between manifolds M^n and N^m. Then the *pullback* $F^*: T^*_{F(x)}N \to T^*_xM$ of F at x is defined by

$$(F^*u)(v) = u(F_*v), \qquad \forall u \in T^*_{F(x)}N$$

for all $v \in T_xM$, where F_* is the push-forward given by F at x.

5.4.2 Vector Fields and Flows

A smooth vector field $v^a(x)$ on M assigns a tangent vector to each tangent space T_xM. If we fix any point $x_0 \in M$, then there is, at least locally, a smooth curve $\gamma: I \to M$ such that $\gamma(0) = x_0$ and such that $\dot{\gamma}(t) = v^a\big(\gamma(t)\big)$ for $t > 0$. The family of curves obtained this way is simply the *integral curves* of the vector field. The *flow* associated with the vector field is the one-parameter diffeomorphism $\phi_t: M \to M$ defined by $\phi_t(x) = \gamma(t)$ where γ is an integral curve to v^a passing through $x \in M$ at $t = 0$, i.e., $\gamma(0) = x$. Although we will not prove it in detail, every smooth vector field has a unique flow associated with it. Notice that $\phi_{t+s} = \phi_s \circ \phi_t$, $\phi_t^{-1} = \phi_{-t}$, and $\phi_0 = \mathrm{id}$ where $\mathrm{id}: M \to M$ is the identity function on M.

5.4.3 Ordinary Partial Differentiation

We want to be able to 'differentiate' tensor fields on a manifold. Let us try to do this using our usual concepts from Euclidean space: Let x^a be a smooth vector field on a smooth manifold M^n. Then, in any local coordinate system, we may consider the n^2 quantities

$$\partial_\beta x^\alpha.$$

Of course these numbers depend upon the coordinate chart used; after all, the intuitive description of $\partial_\beta x^\alpha$ is that it is the rate of change of the scalar field x^α in the direction of the βth coordinate curve. However, perhaps $\partial_\beta x^\alpha$ are simply the components of a tensor $\partial_b x^a$ of type $(1,1)$, as the notation indicates? (But beware that ∂_β is just a symbol denoting $\frac{\partial}{\partial x^\beta}$ which we defined without any consideration of tensors.) That is, if there is another coordinate system with coordinates $x'^1, \dots, x'^n$, then perhaps

$$\partial'_\beta x'^\alpha = \frac{\partial x^\gamma}{\partial x'^\beta}\frac{\partial x'^\alpha}{\partial x^\delta}\partial_\gamma x^\delta?$$

Let us try to show this. Using the chain rule and the vector transformation rule,

$$\partial'_\beta x'^\alpha = \frac{\partial}{\partial x'^\beta}x'^\alpha = \frac{\partial x^\gamma}{\partial x'^\beta}\frac{\partial}{\partial x^\gamma}\left(\frac{\partial x'^\alpha}{\partial x^\delta}x^\delta\right) = \frac{\partial x^\gamma}{\partial x'^\beta}\left(x^\delta\frac{\partial}{\partial x^\gamma}\frac{\partial x'^\alpha}{\partial x^\delta} + \frac{\partial x'^\alpha}{\partial x^\delta}\frac{\partial x^\delta}{\partial x^\gamma}\right) =$$
$$= \frac{\partial x^\gamma}{\partial x'^\beta}\frac{\partial^2 x'^\alpha}{\partial x^\delta \partial x^\gamma}x^\delta + \frac{\partial x^\gamma}{\partial x'^\beta}\frac{\partial x'^\alpha}{\partial x^\delta}\partial_\gamma x^\delta.$$

Had it not been for the first term on the right-hand side, $\partial_a x^b$ *would* have been a tensor of type $(1,1)$. As it is, however, it is *not*. This means that a formulae involving an ordinary partial derivative does not say anything about the actual tensor field or the manifold, but depends upon the coordinate system one happens to be using at the moment. Thus, ordinary partial derivatives are not particularly interesting for us. In what follows, we will define two new types of derivatives, which *are* intrinsic and independent upon coordinate system, namely, the *covariant derivative* and the *Lie derivative*. The covariant derivative is perhaps the most important one for us, but the Lie derivative is a simpler object, and so we start with this one in the next section. First, let us just make

Proposition NN

If $f: M \to \mathbb{R}$ is a scalar field on a manifold $M = M^n$, then the n numbers $\partial_\alpha f$ form the components of a covector.

Proof

Let $x^1, \dots, x^n$ and $x'^1, \dots, x'^n$ be two coordinate systems. Then

$$\partial'_\alpha f = \frac{\partial x^\beta}{\partial x'^\alpha}\partial_\beta f$$

by the chain rule. ∎

5.4.4 The Lie Derivative

Let $\gamma: I \to M$ be a smooth curve on a manifold M. Let $v^a(t) \in T_{\gamma(t)}M$ be a vector field defined along γ. It is tempting to define the derivative of v^a at $t = t_0$ as one moves along γ as the limit

$$\dot{v}^a(t_0) \overset{?}{\coloneqq} \lim_{h\to 0^+} \frac{v^a(t_0+h) \overset{?}{-} v^a(t_0)}{h}.$$

This is perfectly doable in Euclidean space $\mathbb{R}^n$, but not in general. Why is that? Well, $v^a(t_0+h)$ lives in $T_{\gamma(t_0+h)}M$ while $v^a(t_0)$ lives in $T_{\gamma(t_0)}$, and so we cannot subtract them.[65] The Lie derivative is one notion of derivative that *does* makes sense.

Definition NN

Let $v^a(x)$ and $u^a(x)$ be smooth vector fields on M. Then the *Lie derivative* of v^a with respect to u^a at $x \in M$ is

$$\mathcal{L}_u v^a \coloneqq \lim_{t\to 0} \frac{\phi_{(-t)*} v^a\big(\phi_t(x)\big) - v^a(x)}{t}$$

where ϕ_t is the flow associated with u^a.

Notice that $v^a(x) \in T_xM$ and $\phi_{(-t)*} v^a\big(\phi_t(x)\big) \in T_xM$ and so the subtraction is ordinary vector-space subtraction in T_xM. In particular, $\mathcal{L}_u v^a \in T_xM$, and in the abstract index notation, we may write it as $(\mathcal{L}_u v)^a$. Although the definition of the Lie derivative given above has the advantage of motivating the name 'derivative', it is not particularly useful when it comes to practical computations. To remedy this, we first need to consider a new construct.

Recall that a tangent vector $v \in T_xM$ is a *derivation*, and its image on a scalar field $f: M \to \mathbb{R}$ is written $v(f)$. If v is a vector *field*, the number $v(f)$ varies on M, and so it is again a scalar field on M. Thus it makes perfect sense to consider $u\big(v(f)\big)$ where u is another vector field. This can be thought of as applying the operator $u \circ v$ to f, and one might wonder if u and v commute, that is, if $u \circ v = v \circ u$. As usual in many fields of pure and applied mathematics, we can, to this end, introduce the *commutator* of u and v:

Definition NN

Let u^a and v^a be two vector fields defined in a neighbourhood of $x \in M$. Then the map $\mathcal{D}: \mathcal{F}_x \to \mathbb{R}$ defined by

$$\mathcal{D}(f) = [u,v](f) = u\big(v(f)\big) - v\big(u(f)\big), \qquad \forall f \in \mathcal{F}_x$$

is called the *commutator*, or the *Lie bracket*, of u and v.

Proposition NN

The Lie bracket between two vector fields is a vector field.

Proof

We need to show that $[u,v]: \mathcal{F}_x \to \mathbb{R}$ is a derivation at each $x \in M$. Since u and v *are* derivations,

[65] For example, think of $M = S^2 \subset \mathbb{R}^3$. Consider two tangent vectors in two different tangent spaces. What in the world is the sum, or difference, of these? What does it mean that they are 'equal' (that is, that their difference is the 'zero vector' (in what space??)? Remember that it is cheating to use the ambient $\mathbb{R}^3$.

$$\begin{aligned}[u,v](\alpha f+\beta g) &= u\big(v(\alpha f+\beta g)\big)-v\big(u(\alpha f+\beta g)\big)= \\ &= u\big(\alpha v(f)+\beta v(g)\big)-v\big(\alpha u(f)+\beta u(g)\big)= \\ &= \alpha u\big(v(f)\big)+\beta u\big(v(g)\big)-\alpha v\big(u(f)\big)-\beta v\big(u(g)\big)= \\ &= \alpha\Big(u\big(v(f)\big)-v\big(u(f)\big)\Big)+\beta\Big(u\big(v(g)\big)-v\big(u(g)\big)\Big)=\alpha[u,v](f)+\beta[u,v](g)\end{aligned}$$

and (recall that x is a *fixed* point)

$$\begin{aligned}[u,v](f\cdot g) &= u\big(v(f\cdot g)\big)-v\big(u(f\cdot g)\big)= \\ &= u\big(f(x)v(g)+v(f)g(x)\big)-v\big(f(x)u(g)+u(f)g(x)\big)= \\ &= f(x)u\big(v(g)\big)+g(x)u\big(v(f)\big)-f(x)v\big(u(g)\big)-g(x)v\big(u(f)\big)= \\ &= f(x)[u,v](g)+g(x)[u,v](f)\end{aligned}$$

for all $\alpha,\beta\in\mathbb{R}$ and $f,g\in\mathcal{F}_x$. Thus, $[u,v]$ is a derivation at x. ■

Proposition NN

Let u^a, v^a, and w^a be vector fields. Then

(1) $[u,v]=-[v,u]$,

(2) $[u,u]=0$, and

(3) $\big[u,[v,w]\big]+\big[w,[u,v]\big]+\big[v,[w,u]\big]=0$.

Proof

(1) and (2) are immediate, and, in addition, (1) ⇒ (2). The *Jacobi identity* (3) follows from direct computation. Let $f\colon M\to\mathbb{R}$ be arbitrary, and let $z:=[v,w]$. Then

$$\begin{aligned}\big[u,[v,w]\big](f)=[u,z](f) &= u\big(z(f)\big)-z\big(u(f)\big)= \\ &= u\Big(v\big(w(f)\big)-w\big(v(f)\big)\Big)-\Big[v\Big(w\big(u(f)\big)\Big)-w\Big(v\big(u(f)\big)\Big)\Big]= \\ &= u\Big(v\big(w(f)\big)\Big)-u\Big(w\big(v(f)\big)\Big)-v\Big(w\big(u(f)\big)\Big)+w\Big(v\big(u(f)\big)\Big).\end{aligned}$$

By renaming the variables,

$$\begin{aligned}\big[w,[u,v]\big](f) &= w\Big(u\big(v(f)\big)\Big)-w\Big(v\big(u(f)\big)\Big)-u\Big(v\big(w(f)\big)\Big)+v\Big(u\big(w(f)\big)\Big) \quad \text{and} \\ \big[v,[w,u]\big](f) &= v\Big(w\big(u(f)\big)\Big)-v\Big(u\big(w(f)\big)\Big)-w\Big(u\big(v(f)\big)\Big)+u\Big(w\big(v(f)\big)\Big).\end{aligned}$$

The Jacobi identity follows by adding these three equations. ■

Proposition NN

Let $\boldsymbol{\partial_1},\dots,\boldsymbol{\partial_n}$ be the basis vector field in some coordinate patch $(U,\phi_U)\in\mathcal{A}_M$, and let u^a and v^a be two vector fields. Then the components

$$[u,v]^\alpha=u^\beta\partial_\beta v^\alpha-v^\beta\partial_\beta u^\alpha.$$

Proof

For every scalar field $f\colon M\to\mathbb{R}$ of class C^2 or better,

$$[u,v](f) = u\big(v(f)\big) - v\big(u(f)\big) = u(v^\alpha \partial_\alpha f) - v(u^\alpha \partial_\alpha f) = u^\beta \partial_\beta (v^\alpha \partial_\alpha f) - v^\beta \partial_\beta (u^\alpha \partial_\alpha f) =$$
$$= u^\beta \big(v^\alpha \partial_\beta \partial_\alpha f + \partial_\beta v^\alpha \, \partial_\alpha f\big) - v^\beta \big(u^\alpha \partial_\beta \partial_\alpha f + \partial_\beta u^\alpha \, \partial_\alpha f\big) =$$
$$= u^\beta v^\alpha \partial_\beta \partial_\alpha f + u^\beta \partial_\beta v^\alpha \, \partial_\alpha f - v^\beta u^\alpha \partial_\beta \partial_\alpha f - v^\beta \partial_\beta u^\alpha \, \partial_\alpha f.$$

Now,

$$u^\beta v^\alpha \partial_\beta \partial_\alpha f - v^\beta u^\alpha \partial_\beta \partial_\alpha f = u^\beta v^\alpha \partial_\beta \partial_\alpha f - v^\alpha u^\beta \partial_\alpha \partial_\beta f = v^\alpha u^\beta \big(\partial_\beta \partial_\alpha f - \partial_\alpha \partial_\beta f\big) = 0$$

because f is of class C^2. Therefore,

$$[u,v](f) = u^\beta \partial_\beta v^\alpha \, \partial_\alpha f - v^\beta \partial_\beta u^\alpha \, \partial_\alpha f = \big(u^\beta \partial_\beta v^\alpha - v^\beta \partial_\beta u^\alpha\big)\partial_\alpha f$$

and

$$[u,v]^\alpha = u^\beta \partial_\beta v^\alpha - v^\beta \partial_\beta u^\alpha.$$

■

Corollary NN

Let $\boldsymbol{\partial_1, \ldots, \partial_n}$ be the basis vector field in some coordinate patch $(U, \phi_U) \in \mathcal{A}_M$. Then

$$\big[\boldsymbol{\partial_\mu, \partial_\nu}\big]^a = 0.$$

Proof

In the local coordinate patch,

$$\big[\boldsymbol{\partial_\mu, \partial_\nu}\big]^\alpha = \delta_\mu^\beta \partial_\beta \delta_\nu^\alpha - \delta_\nu^\beta \partial_\beta \delta_\mu^\alpha = 0$$

because the Kronecker symbol δ_ν^α is constant for any pair of indices. But a tensor with all components equal to zero in some coordinate system is the zero tensor, and the statement follows. ■

We now come to the main result, showing that the Lie derivative and the Lie bracket *are the same thing!*

Proposition NN

Let $v^a(x)$ and $u^a(x)$ be smooth vector fields on M. Then

$$\mathcal{L}_u v^a = [u,v]^a.$$

Proof

TBW.

5.4.5 The Covariant Derivative

We can obtain a second derivative operator on a general Riemannian manifold by generalising the concept of the covariant derivative discussed for two-surfaces in three-space in Section 4.3.7. Recall that for such an f-surface, we defined the Christoffel symbols in Definition NN, and the expressions for the covariant derivative of a surface-tangential vector field was found in Section 4.3.7.4.

Since the Christoffel symbols can be expressed solely in terms of the coefficients of the first fundamental form, it is not surprising that they can be expressed in terms of the metric tensor.

Proposition NN

Let $\Sigma = \mathbf{r}(D) \subset \mathbb{R}^3$ be a manifold surface with metric tensor g_{ab}. Then the Christoffel symbols

$$\Gamma^{\mu}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}\left(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\alpha g_{\beta\gamma}\right)$$

where the components are given in the coordinate patch $(\Sigma, \mathbf{r}^{-1})$.

Proof

Using tensorial notation, the formulae NN on page NN can be written succinctly as

$$\mathbf{r}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta}\mathbf{r}_\gamma + \left(\mathbf{r}_{\alpha\beta} \cdot \widehat{\mathbf{N}}\right)\widehat{\mathbf{N}}$$

where $x^1 = u$ and $x^2 = v$ and $\alpha, \beta = 1, 2$. Thus,

$$\mathbf{r}_{\alpha\beta} \cdot \mathbf{r}_\delta = \Gamma^{\gamma}_{\alpha\beta}\mathbf{r}_\gamma \cdot \mathbf{r}_\delta = \Gamma^{\gamma}_{\alpha\beta} g_{\gamma\delta}.$$

Since

$$\partial_\gamma g_{\alpha\beta} = \partial_\gamma\left(\mathbf{r}_\alpha \cdot \mathbf{r}_\beta\right) = \mathbf{r}_{\alpha\gamma} \cdot \mathbf{r}_\beta + \mathbf{r}_\alpha \cdot \mathbf{r}_{\beta\gamma} = \Gamma^{\delta}_{\alpha\gamma} g_{\delta\beta} + \Gamma^{\delta}_{\beta\gamma} g_{\delta\alpha}$$

we have

$$\begin{aligned}\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\alpha g_{\beta\gamma} &= \Gamma^{\delta}_{\alpha\gamma} g_{\delta\beta} + \Gamma^{\delta}_{\beta\gamma} g_{\delta\alpha} + \Gamma^{\delta}_{\gamma\beta} g_{\delta\alpha} + \Gamma^{\delta}_{\alpha\beta} g_{\delta\gamma} - \Gamma^{\delta}_{\beta\alpha} g_{\delta\gamma} - \Gamma^{\delta}_{\gamma\alpha} g_{\delta\beta} = \\ &= \Gamma^{\delta}_{\beta\gamma} g_{\delta\alpha} + \Gamma^{\delta}_{\gamma\beta} g_{\delta\alpha} = 2\Gamma^{\delta}_{\beta\gamma} g_{\delta\alpha}\end{aligned}$$

where we have used the symmetry of both the Christoffel symbols and the metric. Thus,

$$\Gamma^{\delta}_{\beta\gamma} g_{\delta\alpha} = \frac{1}{2}\left(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\alpha g_{\beta\gamma}\right)$$

and

$$\Gamma^{\mu}_{\beta\gamma} = \Gamma^{\delta}_{\beta\gamma}\delta^{\mu}_{\delta} = \Gamma^{\delta}_{\beta\gamma} g_{\delta\alpha} g^{\alpha\mu} = \frac{1}{2} g^{\alpha\mu}\left(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\alpha g_{\beta\gamma}\right).$$

■

Using tensorial notation, we can also rewrite the expression (↑) for the covariant derivative of $\mathbf{u}$ with respect to $\mathbf{v}$ much more succinctly as

$$(\nabla_{\mathbf{v}}\mathbf{u})^\alpha = v^\beta\left(\partial_\beta u^\alpha + \Gamma^{\alpha}_{\beta\gamma} u^\gamma\right)$$

(check that!).

Observation NN

The abstract index notation combined with the Einstein summation convention *does* simplify some formulae significantly!

We will now generalise the notion of the covariant derivative to *any* Riemannian manifold. This is extremely straight-forward (at least at first sight), because the expression for the Christoffel symbols in terms of the metric tensor makes sense in any such manifold, as does the formula (↑) for the covariant derivative in terms of the Christoffel symbols.

Definition NN

Let M^n be a Riemannian manifold with metric tensor g. Then, in any coordinate system, the *Christoffel symbols* are

$$\Gamma^{\mu}_{\beta\gamma} \coloneqq \frac{1}{2} g^{\alpha\mu}\left(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\alpha g_{\beta\gamma}\right).$$

If u^a is a vector field on M, then the *covariant derivative of* u^a has components

$$\nabla_\beta u^\alpha \coloneqq \partial_\beta u^\alpha + \Gamma^{\alpha}_{\beta\gamma} u^\gamma.$$

If, in addition, $v^a \in T_x M$ is a vector at $x \in M$, then the *covariant derivative of* u^a *with respect to* v^a *at* x has components

$$(\nabla_v u)^\alpha \coloneqq v^\beta \nabla_\beta u^\alpha = v^\beta\left(\partial_\beta u^\alpha + \Gamma^{\alpha}_{\beta\gamma} u^\gamma\right).$$

Finally, if $\gamma: I \to M$ is a curve on M, then the covariant derivative of $u^a(t) \coloneqq u^a\big(\gamma(t)\big)$ along the curve at time $t \in I$ has components

$$\left(\frac{\nabla u}{dt}\right)^\alpha \coloneqq \left(\nabla_{\dot\gamma(t)} u(t)\right)^\alpha = \dot\gamma(t)^\beta \nabla_\beta u(t)^\alpha.$$

We will use the terms 'covariant derivative' and 'connection' interchangeably. In addition, we will often be 'sloppy' enough to write $\nabla_v u^a$ instead of $(\nabla_v u)^a$.

Let u^a be a vector field. Recall that we in Section 5.4.3 showed that the n^2 numbers $\partial_\beta u^\alpha$, computed in some coordinate system, does *not* form the components of a tensor of type $(1,1)$. However, the next proposition shows that the components $\nabla_\beta u^\alpha \overset{\text{def}}{=} \partial_\beta u^\alpha + \Gamma^{\alpha}_{\beta\gamma} u^\gamma$ *does* indeed form the components of such a tensor! Hence, slightly loosely, one may say that the second term, involving the Christoffel symbols, constitute the *correction* required in order to make a tensorial derivative operator out of the ordinary partial derivative operator.

Proposition NN

If u^a is a vector field on M^n, then the n^2 numbers $\nabla_\beta u^\alpha$ form the components of a mixed tensor of type $(1,1)$.

Proof

TBW.

Corollary NN

If u^a is a vector field on M^n and $v^a \in T_x M$ is a vector at $x \in M$, then the n numbers $(\nabla_v u)^\alpha$ form the components of a contravariant vector at x, written $(\nabla_v u)^a \in T_x M$. If v^a is a vector *field* on M, then $(\nabla_v u)^a$ is too a vector field on M.

Corollary NN

The Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ does *not* form the components of a tensor of type $(1,2)$.

Proof

The definition (↑) implies

$$\nabla_\beta u^\alpha - \Gamma^\alpha_{\beta\gamma} u^\gamma = \partial_\beta u^\alpha.$$

If $\Gamma^\alpha_{\beta\gamma}$ would be a tensor of type $(1,2)$, then the contraction $\Gamma^\alpha_{\beta\gamma} u^\gamma$ with the tensor u^c would be a tensor of type $(1,1)$, and so would the vector-space difference $\nabla_\beta u^\alpha - \Gamma^\alpha_{\beta\gamma} u^\gamma$. Thus, $\partial_\beta u^\alpha$ *would* be a tensor of type $(1,1)$. ■

It should be noted that different authors use slightly different notation. For instance, (Frankel, 2004) reserves the word 'covariant derivative' for the vector $(\nabla_v u)^a = v^b \nabla_b u^a$, while (Wald, 1984) uses it to denote the mixed tensor $\nabla_b u^a$, as we do. It should also be noted that, in more general treatments of differential geometry, the word 'covariant derivative', or '/affine/ connection', is used to denote a wide *class* of differential operators that are required to satisfy a number of axioms, whereas we will only be concerned with the most common connection, namely, the one that we defined above, in terms of the metric tensor. This connection is known specifically as the *Levi-Civita* connection and is an example of a *metric* connection.

We consider two basic properties of the (Levi-Civita) covariant derivative, namely, linearity and the Leibniz property.

Proposition NN

Let ∇ be the covariant derivative on a Riemannian manifold. Then

(1) $\nabla_b(\alpha u^a + \beta v^a) = \alpha \nabla_b u^a + \beta \nabla_b v^a$ and

(2) $\nabla_b(f u^a) = (\partial_b f) u^a + f \nabla_b u^a$

for all smooth vector fields u^a and v^a, constants $\alpha, \beta \in \mathbb{R}$, and scalar fields f.

Proof

(1) $\nabla_b(\alpha u^a + \beta v^a) \overset{\text{def}}{=} \partial_b(\alpha u^a + \beta v^a) + \Gamma^a_{bc}(\alpha u^c + \beta v^c) = \alpha(\partial_b u^a + \Gamma^a_{bc} u^c) + \beta(\partial_b v^a + \Gamma^a_{bc} + v^c) = \alpha \nabla_b u^a + \beta \nabla_b v^a$ and

(2) $\nabla_b(f u^a) \overset{\text{def}}{=} \partial_b(f u^a) + \Gamma^a_{bc}(f u^c) = (\partial_b f) u^a + f \partial_b u^a + f \Gamma^a_{bc} u^c = (\partial_b f) u^a + f \nabla_b u^a.$ ■

Let us then consider a few basic properties of the Levi-Civita covariant derivative along a vector.

Proposition NN

Let ∇ be the covariant derivative on a Riemannian manifold M, and let $X, Y \in T_x M$. Then

(1) $\nabla_X(\alpha u^a + \beta v^a) = \alpha \nabla_X u^a + \beta \nabla_X v^a$,

(2) $\nabla_{(\alpha X + \beta Y)} u^a = \alpha \nabla_X u^a + \beta \nabla_Y u^a$, and

(3) $\nabla_X(f u^a) = X(f) u^a + f \nabla_X u^a$

for all smooth vector fields u^a and v^a, constants $\alpha, \beta \in \mathbb{R}$, and scalar fields f.

Proof

(1) $\nabla_X(\alpha u^a + \beta v^a) \overset{\text{def}}{\equiv} X^b \nabla_b(\alpha u^a + \beta v^a) \underset{\text{last prop.}}{=} X^b(\alpha \nabla_b u^a + \beta \nabla_b v^a) = \alpha X^b \nabla_b u^a + \beta X^b \nabla_b v^a = \alpha \nabla_X u^a + \beta \nabla_X u^a$,

(2) $\nabla_{(\alpha X + \beta Y)} u^a \overset{\text{def}}{\equiv} \big(\alpha X^b + \beta Y^b\big)\nabla_b u^a = \alpha X^b \nabla_b u^a + \beta Y^b \nabla_b u^a = \alpha \nabla_X u^a + \beta \nabla_Y u^a$, and

(3) $\nabla_X(f u^a) \overset{\text{def}}{\equiv} X^b \nabla_b(f u^a) \underset{\text{last prop.}}{=} X^b\big((\partial_b f) u^a + f \nabla_b u^a\big) = X^b(\partial_b f) u^a + X^b f \nabla_b u^a = X(f) u^a + f \nabla_X u^a$. ■

Any map taking a vector field u^a and a vector X to a vector that satisfies the requirements (1)-(3) in Proposition NN is called a 'covariant derivative', or a 'connection', using the generalised sense of the word (as used in Frankel). Thus, the Levi-Civita connection *is* a connection even in this sense. It can be shown that any connection can be written in the form (↑) using *some* set of n^3 numbers $\Gamma^\alpha_{\beta\gamma}$ that transforms under a certain way under a change of basis. These are called the *coefficients of the connection*. Thus, the Levi-Civita is the connection in which the coefficients are the usual Christoffel symbols.

So far we have defined the covariant derivative for vector fields. We may also define it for scalar fields:

Definition NN

Let f be a scalar field on a manifold M. Then, let

$$\nabla_a f \coloneqq \partial_a f$$

and, if $v \in T_x M$, let

$$\nabla_v f \coloneqq v^a \nabla_a f = v^a \partial_a f = v(f).$$

Besides being natural in its own right, this has the important benefit of making the Leibniz property of the covariant derivative more natural. Indeed, now

$$\nabla_b(f u^a) = (\nabla_b f) u^a + f \nabla_b u^a$$

and

$$\nabla_X(f u^a) = (\nabla_X f) u^a + f \nabla_X u^a.$$

We can extend the covariant derivative to work with any tensor or arbitrary type (k, l) by demanding that the Leibniz property always holds. For example, if u^a is a vector field and v_a is a covector field, then $u^a v_a$ is a scalar field, and

$$\nabla_b(u^a v_a) \underset{\text{Leibniz}}{=} (\nabla_b u^a) v_a + u^a \nabla_b v_a$$

Therefore,

$$\begin{aligned} u^a \nabla_b v_a = \nabla_b(u^a v_a) - (\nabla_b u^a) v_a &\overset{\text{defs}}{=} \partial_b(u^a v_a) - (\partial_b u^a + \Gamma^a_{bc} u^c) v_a = \\ &= (\partial_b u^a) v_a + u^a \partial_b v_a - (\partial_b u^a) v_a - \Gamma^a_{bc} u^c v_a = u^a \partial_b v_a - \Gamma^a_{bc} u^c v_a = \\ &= u^a(\partial_b v_a - \Gamma^c_{ba} v_c) \end{aligned}$$

should hold for every vector field u^a. Hence

$$\nabla_b v_a = \partial_b v_a - \Gamma^c_{ba} v_c$$

is the expression for the covariant derivative of a covector field. Proceeding in a similar manner, one can show that

$$\nabla_c T_{ab} = \partial_c T_{ab} - \Gamma^d_{ac} T_{db} - \Gamma^d_{bc} T_{ad}$$

The Levi-Civita connection is a *metric* connection. The property proven below is the defining characteristic of such a connection.

Proposition NN

Let ∇ be the covariant derivative on a Riemannian manifold with metric g. Then

$$\nabla_c g_{ab} = 0.$$

Proof

$$\begin{aligned}\nabla_c g_{ab} &= \partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} = \\ &= \partial_c g_{ab} - \frac{1}{2} g^{md}(\partial_c g_{ma} + \partial_a g_{cm} - \partial_m g_{ac}) g_{db} - \\ &- \frac{1}{2} g^{md}(\partial_c g_{mb} + \partial_b g_{cm} - \partial_m g_{bc}) g_{ad} = \\ &= \partial_c g_{ab} - \frac{1}{2}\delta^m_b(\partial_c g_{ma} + \partial_a g_{cm} - \partial_m g_{ac}) - \frac{1}{2}\delta^m_a(\partial_c g_{mb} + \partial_b g_{cm} - \partial_m g_{bc}) = \\ &= \partial_c g_{ab} - \frac{1}{2}(\partial_c g_{ba} + \partial_a g_{cb} - \partial_b g_{ac}) - \frac{1}{2}(\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc}) = 0\end{aligned}$$

because the metric tensor is symmetric. ■

It is natural to ask whether or not the covariant derivative is symmetric in the sense that

$$\nabla_u v^a \stackrel{?}{=} \nabla_v u^a$$

for all vector fields u^a and v^a. Thus, we investigate

$$\begin{aligned}\nabla_u v^a - \nabla_v u^a &= u^b(\partial_b v^a + \Gamma^a_{bc} v^c) - v^b(\partial_b u^a + \Gamma^a_{bc} u^c) = \\ &= u^b \partial_b v^a + u^b \Gamma^a_{bc} v^c - v^b \partial_b u^a - v^c \Gamma^a_{cb} u^b = u^b \partial_b v^a - v^b \partial_b u^a = [u, v]^a\end{aligned}$$

because the Christoffel symbols are symmetric. If we had used a general connection, which generally has non-symmetric connection coefficients, we would end up with the so-called *torsion tensor*[66]

$$\tau^a_{bc} \coloneqq \Gamma^a_{bc} - \Gamma^a_{cb}$$

generally being non-zero, and then

$$\nabla_u v^a - \nabla_v u^a - [u, v]^a = \tau^a_{bc} u^b v^c.$$

In the Levi-Civita connection, the coefficients (the Christoffel symbols) *are* symmetric, and so the torsion tensor vanishes identically. Consequently,

$$\nabla_u v^a - \nabla_v u^a - [u, v]^a = 0.$$

[66] It follows from the transformation law of the Christoffel symbols that the right-hand side *is* a tensor of the indicated type.

A connection having this property ($\tau^a_{bc} = 0$) is called *symmetric*. Hence, the Levi-Civita connection is a *symmetric* metric connection.

5.4.6 Examples of Christoffel Symbols

Since the Christoffel symbols play such an important role when it comes to the metric properties of manifolds, we will spend some time deriving the Christoffel symbols for a number of common manifolds.

Example NN

Let $M^n = \mathbb{R}^n$ be Euclidean space with its standard, Cartesian, coordinate system. Then

$$\Gamma^\alpha_{\beta\gamma} \equiv 0$$

because the metric is constant.

Example NN

Consider the cylinder of radius $a > 0$,

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = a^2\},$$

with its usual coordinate system (φ, z). Then

$$g_{\alpha\beta} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g^{\alpha\beta} = \begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\Gamma^\alpha_{\beta\gamma} \equiv 0$$

because the metric is constant.

Example NN

Consider the 2-sphere of radius $a > 0$,

$$S^2_a \coloneqq \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = a\}$$

with its usual coordinate system (θ, φ). Now

$$g_{\alpha\beta} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{pmatrix}, \qquad g^{\alpha\beta} = \begin{pmatrix} a^{-2} & 0 \\ 0 & a^{-2} \csc^2\theta \end{pmatrix}$$

(recall that $g^{\alpha\beta}$, by definition, is the matrix inverse of $g_{\alpha\beta}$). Then, some manual effort yields

$$\Gamma^1_{\alpha\beta} = \frac{1}{2} g^{\delta 1}\left(\partial_\beta g_{\delta\alpha} + \partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta}\right) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}\sin 2\theta \end{pmatrix}$$

and

$$\Gamma^2_{\alpha\beta} = \frac{1}{2} g^{\delta 2}\left(\partial_\beta g_{\delta\alpha} + \partial_\alpha g_{\beta\delta} - \partial_\delta g_{\alpha\beta}\right) = \begin{pmatrix} 0 & \cot\theta \\ \cot\theta & 0 \end{pmatrix}.$$

5.4.7 Curvature

Let

A.1 Linear Algebra

In this appendix we will discuss one of the prerequisites for this book, namely, linear algebra. We do this for two reasons. First, we wish to introduce the notation used for vectors, which might not be familiar to some readers. Second, the transformation properties of vectors and linear transformations are used when we introduce tensors in Section 5.1.

A.1.1 Vectors

Consider the vector space $\mathbb{R}^n$ with its standard basis

$$\begin{aligned}\mathbf{e}_1 &= (1, 0, 0, \dots, 0)\\ \mathbf{e}_2 &= (0, 1, 0, \dots, 0)\\ &\vdots\\ \mathbf{e}_n &= (0, 0, \dots, 0, 1).\end{aligned}$$

Define the (formal) *row matrix* of basis vectors

$$\underline{\mathbf{e}} \coloneqq (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n).$$

Now, consider a vector $\mathbf{v} = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$. We may write

$$\mathbf{v} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + \cdots + x^n\mathbf{e}_n = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n)\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} = \underline{\mathbf{e}}\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} = \underline{\mathbf{e}}X_{\underline{\mathbf{e}}}$$

where

$$X_{\underline{\mathbf{e}}} \coloneqq \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$$

is called the *coordinate matrix* for $\mathbf{v}$ in the basis $\underline{\mathbf{e}}$. Now, if the n vectors

$$\mathbf{f}_1 = \underline{\mathbf{e}}\begin{pmatrix} f_1^1 \\ f_1^2 \\ \vdots \\ f_1^n \end{pmatrix}, \qquad \mathbf{f}_2 = \underline{\mathbf{e}}\begin{pmatrix} f_2^1 \\ f_2^2 \\ \vdots \\ f_2^n \end{pmatrix}, \qquad \dots, \qquad \mathbf{f}_n = \underline{\mathbf{e}}\begin{pmatrix} f_n^1 \\ f_n^2 \\ \vdots \\ f_n^n \end{pmatrix}$$

are linearly independent, then they form a new basis for $\mathbb{R}^n$. We similarly define

$$\underline{\mathbf{f}} \coloneqq (\mathbf{f}_1 \quad \mathbf{f}_2 \quad \cdots \quad \mathbf{f}_n)$$

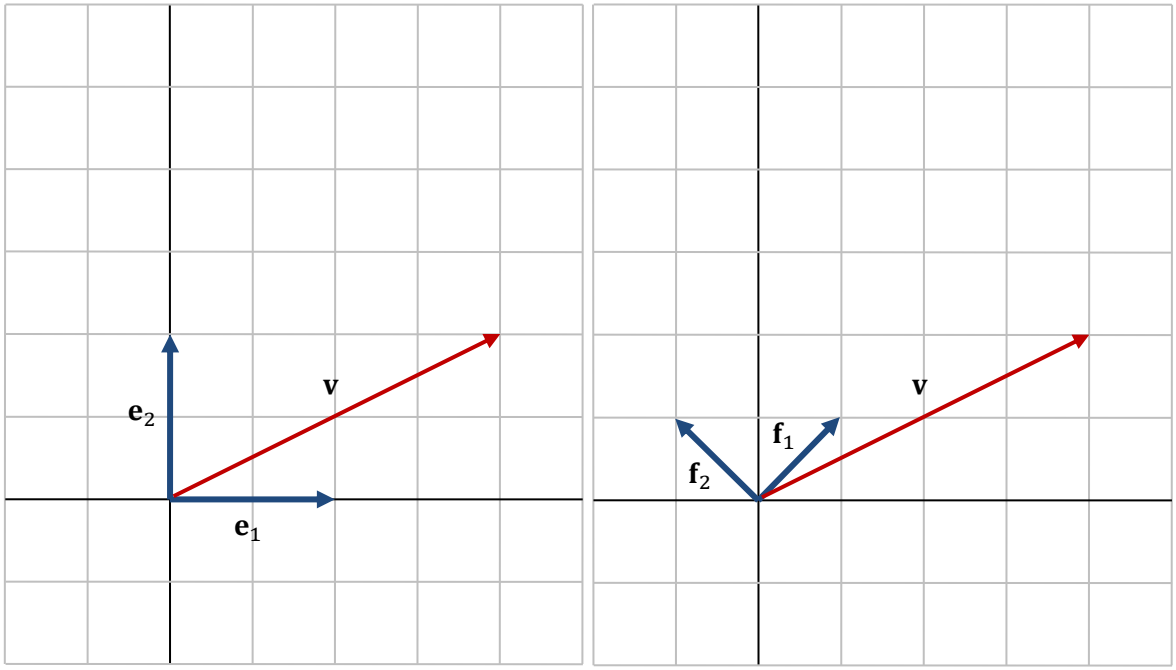
and we may refer to the basis as the $\underline{\mathbf{e}}$ and the $\underline{\mathbf{f}}$ bases, respectively. If $\mathbf{v} = (x^1, x^2, \dots, x^n)$ has coordinates $y^1, y^2, \dots, y^n$ relative to $\underline{\mathbf{f}}$, then we may write

$$\mathbf{v} = \underline{\mathbf{e}}\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} = \underline{\mathbf{f}}\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{pmatrix}$$

which (obviously) is a short-hand notation for

$$\mathbf{v} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + \cdots + x^n\mathbf{e}_n = y^1\mathbf{f}_1 + y^2\mathbf{f}_2 + \cdots + y^n\mathbf{f}_n.$$

For example, let $n = 2$ and consider the vector $\mathbf{v} = (2, 1)$. Then, if $\mathbf{f}_1 = \frac{1}{2}(1, 1)$ and $\mathbf{f}_2 = \frac{1}{2}(-1, 1)$, we have the situation shown in Figure 55.



$$\mathbf{v} = 2\mathbf{e}_1 + \mathbf{e}_2 = \\ = (\mathbf{e}_1 \quad \mathbf{e}_2)\begin{pmatrix}2\\1\end{pmatrix} = \underline{\mathbf{e}}\begin{pmatrix}2\\1\end{pmatrix}$$

$$\mathbf{v} = 3\mathbf{f}_1 - \mathbf{f}_2 = \\ = (\mathbf{f}_1 \quad \mathbf{f}_2)\begin{pmatrix}3\\-1\end{pmatrix} = \underline{\mathbf{f}}\begin{pmatrix}3\\-1\end{pmatrix}$$

Figure 55. A vector has different coordinates in different bases.

We now wish to derive the relationship between the coordinates, or 'components', of a vector in different bases. Suppose that

$$\mathbf{v} = \underline{\mathbf{e}}\begin{pmatrix}x^1\\x^2\\\vdots\\x^n\end{pmatrix}$$

and that we introduce a new basis by (↑). We wish to find the coordinates of $\mathbf{v}$ in this new basis. In other words, we wish to find n unique scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ such that

$$\mathbf{v} = \lambda_1\mathbf{f}_1 + \lambda_2\mathbf{f}_2 + \cdots + \lambda_n\mathbf{f}_n.$$

But this is

$$\lambda_1\underline{\mathbf{e}}\begin{pmatrix}f_1^1\\f_1^2\\\vdots\\f_1^n\end{pmatrix} + \lambda_2\underline{\mathbf{e}}\begin{pmatrix}f_2^1\\f_2^2\\\vdots\\f_2^n\end{pmatrix} + \cdots + \lambda_n\underline{\mathbf{e}}\begin{pmatrix}f_n^1\\f_n^2\\\vdots\\f_n^n\end{pmatrix} = \underline{\mathbf{e}}\begin{pmatrix}x^1\\x^2\\\vdots\\x^n\end{pmatrix}$$

which is equivalent to the matrix equation (that is, system of linear equations)

$$\begin{pmatrix}f_1^1 & f_2^1 & \cdots & f_n^1\\f_1^2 & f_2^2 & \cdots & f_n^2\\\vdots & \vdots & \ddots & \vdots\\f_1^n & f_2^n & \cdots & f_n^n\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\\\vdots\\\lambda_n\end{pmatrix} = \begin{pmatrix}x^1\\x^2\\\vdots\\x^n\end{pmatrix}.$$

Denote the $n \times n$ matrix in (↑) by T. Then, clearly, the *columns of T are the coordinates of the new basis vectors expressed in the old basis*, and $\det T \neq 0$. Thus, we have shown that, as soon as one introduces a new basis, one can collect all information about the change of basis in the so-called *change of basis matrix T*. And then, if

$$\mathbf{v} = \underline{\mathbf{e}} X_{\underline{\mathbf{e}}} = \underline{\mathbf{f}} X_{\underline{\mathbf{f}}},$$

we have the result

$$\boxed{X_{\underline{\mathbf{e}}} = T X_{\underline{\mathbf{f}}}.}$$

A.1.2 Linear Transformations

Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Given a basis $\underline{\mathbf{e}}$ in $\mathbb{R}^n$, there exists a unique matrix $A_{\underline{\mathbf{e}}}$ such that

$$F(\mathbf{v}) = \underline{\mathbf{e}} A_{\underline{\mathbf{e}}} X_{\underline{\mathbf{e}}}, \qquad \forall \mathbf{v} = \underline{\mathbf{e}} X_{\underline{\mathbf{e}}} \in \mathbb{R}^n.$$

Obviously, the ith column of $A_{\underline{\mathbf{e}}}$ consists of the coordinates (in the basis $\underline{\mathbf{e}}$) of the image $T(\mathbf{e}_i)$ of the ith basis vector $\mathbf{e}_i$. Now change to a new basis $\underline{\mathbf{f}}$ according to (↑), and let T be the change of basis matrix. Assume that F has matrix $A_{\underline{\mathbf{f}}}$ relative to $\underline{\mathbf{f}}$, i.e.,

$$F(\mathbf{v}) = \underline{\mathbf{f}} A_{\underline{\mathbf{f}}} X_{\underline{\mathbf{f}}}, \qquad \forall \mathbf{v} = \underline{\mathbf{f}} X_{\underline{\mathbf{f}}} \in \mathbb{R}^n.$$

We wish to find the relation between the matrices $A_{\underline{\mathbf{e}}}$ and $A_{\underline{\mathbf{f}}}$. To this end, pick a single

$$\mathbf{v} = \underline{\mathbf{e}} X_{\underline{\mathbf{e}}} = \underline{\mathbf{f}} X_{\underline{\mathbf{f}}}.$$

Then

$$X_{\underline{\mathbf{e}}} = T X_{\underline{\mathbf{f}}} \quad \text{and} \quad A_{\underline{\mathbf{e}}} X_{\underline{\mathbf{e}}} = T\left(A_{\underline{\mathbf{f}}} X_{\underline{\mathbf{f}}}\right)$$

and, consequently,

$$A_{\underline{\mathbf{e}}} X_{\underline{\mathbf{e}}} = T A_{\underline{\mathbf{f}}} X_{\underline{\mathbf{f}}} \Leftrightarrow T^{-1} A_{\underline{\mathbf{e}}} X_{\underline{\mathbf{e}}} = A_{\underline{\mathbf{f}}} X_{\underline{\mathbf{f}}} \Leftrightarrow T^{-1} A_{\underline{\mathbf{e}}} T X_{\underline{\mathbf{f}}} = A_{\underline{\mathbf{f}}} X_{\underline{\mathbf{f}}}.$$

This being so for *every* $\mathbf{v}$, we have

$$F(\mathbf{v}) = A_{\underline{\mathbf{f}}} X_{\underline{\mathbf{f}}} = T^{-1} A_{\underline{\mathbf{e}}} T X_{\underline{\mathbf{f}}}, \qquad \forall \mathbf{v} = \underline{\mathbf{f}} X_{\underline{\mathbf{f}}} \in \mathbb{R}^n.$$

Uniqueness of the transformation matrix yields

$$\boxed{A_{\underline{\mathbf{f}}} = T^{-1} A_{\underline{\mathbf{e}}} T}$$

which is the relation between the matrices of F in the old and the new basis.

A.2 The Two-Body Gravitational-Attraction ODE

In Section 1.4.1 we encountered the initial-value problem

$$r'' + kr^{-2} = 0, \qquad \begin{cases} r(0) = r_0 \\ r'(0) = 0 \end{cases} \quad (k > 0)$$

while making inquiries about the distance $r(t)$ between two gravitationally attracted bodies at time $t > 0$. In this appendix, we will solve this using exact methods, but we will find that the solution 'most likely' cannot be expressed using elementary functions. First we integrate once, and we will find that the resulting first-order ODE is separable.

$$\ddot{r} + \frac{k}{r^2} = 0 \underset{\dot{r}\neq 0}{\Longleftrightarrow} \ddot{r}\cdot\dot{r} + \frac{k}{r^2}\cdot\dot{r} = 0 \Leftrightarrow \frac{d}{dt}\left[\frac{1}{2}\dot{r}^2 - \frac{k}{r}\right] = 0 \underset{\text{IC}}{\Leftrightarrow} \frac{1}{2}\dot{r}^2 - \frac{k}{r} = -\frac{k}{r_0} \underset{\dot{r}<0}{\Longleftrightarrow} \dot{r} = -\sqrt{2k}\sqrt{\frac{1}{r} - \frac{1}{r_0}} \Leftrightarrow$$

$$\Leftrightarrow \frac{-1}{\sqrt{2k}\sqrt{\frac{1}{r} - \frac{1}{r_0}}}\dot{r} = 1 \Leftrightarrow -\frac{1}{\sqrt{2k}}\frac{d}{dt}\left[\int \left(\frac{1}{r} - \frac{1}{r_0}\right)^{-\frac{1}{2}} dr\right] = 1$$

where

$$\int \left(\frac{1}{r} - \frac{1}{r_0}\right)^{-\frac{1}{2}} dr = \int \frac{dr}{\sqrt{\frac{1}{r} - \frac{1}{r_0}}} = \begin{bmatrix} r = q^{-1} \\ dr = -q^{-2}dq \end{bmatrix} = \int \frac{-dq}{q^2\sqrt{q - r_0^{-1}}} = \begin{bmatrix} s = \sqrt{q - r_0^{-1}} \\ s^2 = q - r_0^{-1} \\ 2sds = dq \end{bmatrix} =$$

$$= \int \frac{-2sds}{(s^2 + r_0^{-1})^2 s} = -2\int \frac{ds}{(s^2 + r_0^{-1})^2}.$$

This last integral is a standard problem from elementary calculus. Let

$$I_2 := \int \frac{ds}{(s^2 + d^{-1})^2}$$

and

$$I_1 := \int \frac{ds}{s^2 + r_0^{-1}} = \int 1\cdot\frac{ds}{s^2 + r_0^{-1}} = \frac{s}{s^2 + r_0^{-1}} + 2\int \frac{s^2 ds}{(s^2 + r_0^{-1})^2} =$$

$$= \frac{s}{s^2 + r_0^{-1}} + 2\int \frac{s^2 + r_0^{-1} - r_0^{-1}}{(s^2 + r_0^{-1})^2} ds = \frac{s}{s^2 + r_0^{-1}} + 2I_1 - 2r_0^{-1}I_2$$

so that

$$I_2 = \frac{r_0}{2}I_1 + \frac{sr_0}{2(s^2 + r_0^{-1})}$$

where

$$I_1 = \int \frac{ds}{s^2 + r_0^{-1}} = d\int \frac{ds}{\left(s\sqrt{r_0}\right)^2 + 1} = \sqrt{d}\arctan\left(s\sqrt{r_0}\right) + \text{const.}$$

Consequently,

$$I_2 = \frac{r_0^{3/2}}{2}\arctan\left(s\sqrt{r_0}\right) + \frac{sr_0}{2(s^2 + r_0^{-1})} + \text{const.}$$

and

$$\int \left(\frac{1}{r} - \frac{1}{r_0}\right)^{-\frac{1}{2}} dr = -2\int \frac{ds}{(s^2 + r_0^{-1})^2} = -2I_2 = -r_0^{3/2}\arctan\left(s\sqrt{r_0}\right) - \frac{sr_0}{(s^2 + r_0^{-1})} + \text{const.} =$$
$$= -r_0^{3/2}\arctan\left(\sqrt{q - r_0^{-1}}\sqrt{r_0}\right) - \frac{\sqrt{q - r_0^{-1}}r_0}{(q - r_0^{-1} + r_0^{-1})} + \text{const.} =$$
$$= -r_0^{3/2}\arctan\left(\sqrt{qr_0 - 1}\right) - \frac{r_0}{q}\sqrt{q - r_0^{-1}} + \text{const.} =$$
$$= -r_0^{3/2}\arctan\left(\sqrt{r^{-1}r_0 - 1}\right) - r_0 r\sqrt{r^{-1} - r_0^{-1}} + \text{const.}$$

Therefore,

$$\ddot{r} + \frac{k}{r^2} = 0 \Leftrightarrow -\frac{1}{\sqrt{2k}}\frac{d}{dt}\left[-r_0^{3/2}\arctan\left(\sqrt{r^{-1}r_0 - 1}\right) - r_0 r\sqrt{r^{-1} - r_0^{-1}}\right] = 1 \Leftrightarrow$$
$$\Leftrightarrow \frac{d}{dt}\left[r_0^{3/2}\arctan\left(\sqrt{r^{-1}r_0 - 1}\right) + r_0 r\sqrt{r^{-1} - r_0^{-1}}\right] =$$
$$= \sqrt{2k} \underset{\text{IC}}{\Leftrightarrow} r_0^{3/2}\arctan\left(\sqrt{r^{-1}r_0 - 1}\right) + r_0 r\sqrt{r^{-1} - r_0^{-1}} = \sqrt{2k}\cdot t \Leftrightarrow$$
$$\Leftrightarrow r_0^{3/2}\arctan\left(\sqrt{r^{-1}r_0 - 1}\right) + \sqrt{r_0}r\sqrt{r^{-1}r_0 - 1} = \sqrt{2k}\cdot t.$$

For simplicity, put

$$p \coloneqq \sqrt{r^{-1}r_0 - 1}.$$

Recall that $r \in [0, r_0[$. Clearly, $p \in [0, \infty[$ and the expression $r \mapsto p$ above is a bijection between these intervals. The inverse is

$$r = \frac{r_0}{p^2 + 1}.$$

Hence,

$$\ddot{r} + \frac{k}{r^2} = 0 \Leftrightarrow r_0^{3/2}\arctan p + \sqrt{r_0}\cdot\frac{r_0}{p^2 + 1}\cdot p = \sqrt{2k}\cdot t \Leftrightarrow r_0^{3/2}\left(\arctan p + \frac{p}{p^2 + 1}\right) = \sqrt{2k}\cdot t.$$

Unfortunately, the function

$$p \mapsto F(p) \coloneqq \arctan p + \frac{p}{p^2 + 1}$$

is very hard to invert. But since

$$\frac{dF}{dp} = \frac{2}{(1 + p^2)^2} > 0, \qquad \forall p \in [0, \infty[,$$

F is strictly increasing and thus F^{-1} exists. Since $F\colon [0, \infty[\to \left[0, \frac{\pi}{2}\right[$ we have $F^{-1}\colon \left[0, \frac{\pi}{2}\right[\to [0, \infty[$. Let $\mathcal{G} \coloneqq F^{-1}$. It is trivial that $\frac{d\mathcal{G}}{dx} > 0, \forall x \in \left[0, \frac{\pi}{2}\right[$ and that $\mathcal{G}(x) \to \infty$ as $x \to \frac{\pi}{2}$. Now,

$$\ddot{r}+\frac{k}{r^2}=0 \Leftrightarrow r_0^{3/2}F(p)=\sqrt{2k}\cdot t \Leftrightarrow F(p)=r_0^{-3/2}\sqrt{2k}\cdot t \Leftrightarrow p=\mathcal{G}\left(r_0^{-3/2}\sqrt{2k}\cdot t\right) \Leftrightarrow$$
$$\Leftrightarrow \sqrt{r^{-1}r_0-1}=\mathcal{G}\left(r_0^{-3/2}\sqrt{2k}\cdot t\right) \Leftrightarrow r=\frac{r_0}{\left(\mathcal{G}\left(r_0^{-3/2}\sqrt{2k}\cdot t\right)\right)^2+1}.$$

We have therefore arrived at the solution

$$r(t)=\frac{r_0}{\left(\mathcal{G}\left(r_0^{-3/2}\sqrt{2k}\cdot t\right)\right)^2+1}$$

where $\mathcal{G}\colon \left[0,\frac{\pi}{2}\right[\to [0,\infty[$ is the inverse of $p \mapsto \arctan p+\frac{p}{p^2+1}$.

Notice that $r(t)\to 0$ precisely as $\mathcal{G}\left(r_0^{-3/2}\sqrt{2k}\cdot t\right)\to\infty$, i.e., as $r_0^{-3/2}\sqrt{2k}\cdot t\to\frac{\pi}{2}$, i.e., as

$$t\to\frac{\pi}{2\sqrt{2k}}r_0^{3/2}=:t_{\text{collision}}.$$

Thus, even though the solution $t\mapsto r(t)$ to the ODE (most likely) cannot be expressed using elementary functions, the 'collision time' can. [From a purely mathematical point of view, the 'collision time' may be defined as the (smallest) number $t_{\text{collision}}>0$ such that $r(t)\to 0$ as $t\to t_{\text{collision}}$.]

A plot of $r(t)$ versus t is given in Figure 9, which is obtained by numerical integration for particular values of r_0 and k.

A.3 Index

Index goes here.

A.4 Figure Index

A.5 Bibliography

Darling, R. (1994). *Differential Forms and Connections.* Cambridge University Press.

d'Inverno, R. (1992). *Introducing Einstein's Relativity.* Oxford University Press.

Einstein, A. (1905). Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig? *Annalen der Physik, 323*(13), 639-641.

Ellis, G. F., & van Elst, H. (1998). Cosmological Models. *Cargèse Lectures.*

Frankel, T. (2004). *The Geometry of Physics.* Cambridge University Press.

Hafele, J., & Keating, R. E. (1972, Jul. 14). Around-the-World Atomic Clocks: Observed Relativistic Time Gains. *Science*, pp. 168-170.

Hafele, J., & Keating, R. E. (1972, Jul. 14). Around-the-World Atomic Clocks: Predicted Relativistic Time Gains. *Science*, pp. 166-168.

Ludvigsen, M. (1999). *General Relativity - A Geometric Approach.* Cambridge University Press.

Rossi, B., & Hall, D. B. (1941, Feb. 1941). Variation of the Rate of Decay of Mesotrons with Momentum. *The Physical Review, 59*(3), pp. 223-228.

Saff, E., & Snider, A. (2003). *Fundamentals of Complex Analysis with Applications to Engineering and Science.* Pearson Education.

Spivak, M. (1965). *Calculus on Manifolds - A Modern Approach to Classical Theorems of Advanced Calculus.* Westview Press.

Synge, J., & Schild, A. (1949). *Tensor Calculus.* Dover Publications.

Taylor, J. R. (2005). *Classical Mechanics.* University Science Books.

Tipler, P. A., & Llewellyn, R. A. (2008). *Modern Physics* (5th ed.). W. H. Freeman and Company.

Wald, R. M. (1984). *General Relativity.* The University of Chicago Press.

Walter, A. S. (2008). *Partial Differential Equations - An Introduction.* Wiley.

Weinberg, S. (2008). *Cosmology.* Oxford University Press.

Wikipedia contributors. (2011, April 2). *Muon*. Retrieved April 17, 2011, from Wikipedia, The Free Encyclopedia: http://en.wikipedia.org/w/index.php?title=Muon&oldid=422036751